CALTECH
Control & Dynamical Systems

Invariant Manifolds and Transport in the Three-Body Problem

Shane D. Ross

Control and Dynamical Systems
California Institute of Technology

Classical N-Body Systems and Applications
University of Warwick, 14-20 April 2002

Acknowledgements

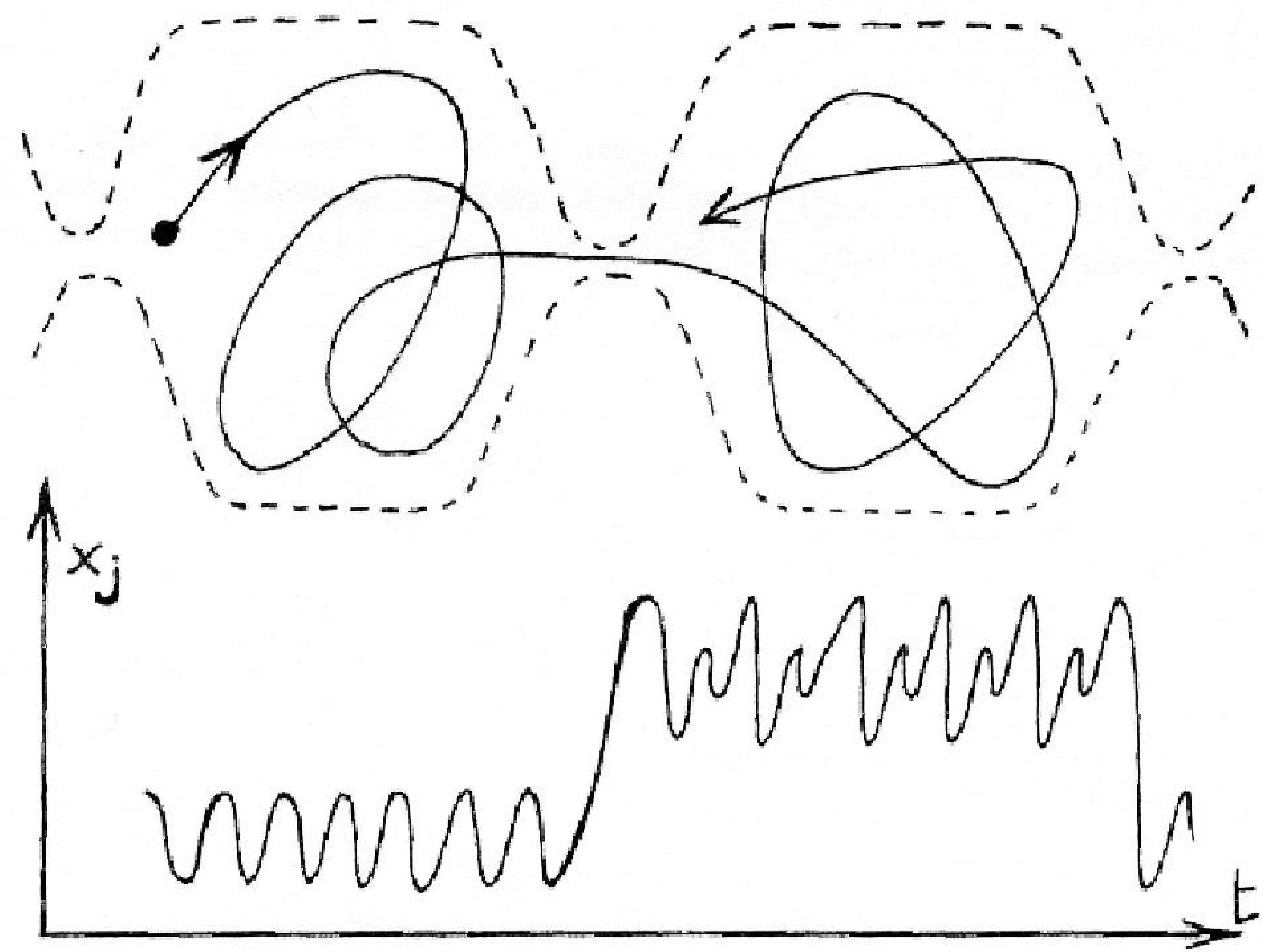
- W. Koon, M. Lo, J. Marsden
- G. Gómez, J. Masdemont
- C. Jaffé, D. Farrelly, T. Uzer
- K. Howell, B. Barden, R. Wilson
- C. Simó, J. Llibre, R. Martinez
- E. Belbruno, B. Marsden, J. Miller
- L. Petzold, S. Radu
- H. Poincaré, J. Moser
- C. Conley, R. McGehee

Outline

- ***Transport theory***
  - Time-independent N-body Hamiltonian systems
  - Examples:
    - ionization of Rydberg atoms
    - restricted three-body problem

Chaotic Dynamics

Transport through a “bottleneck” in phase space; intermittency



Transport Theory

- ***Chaotic dynamics***
  $\Longrightarrow$ ***statistical methods***
- ***Transport theory***
  - Ensembles of phase space trajectories
    - How long (or likely) to move from one region to another?
    - Determine transition probabilities, correlation functions
  - Applications:
    - Atomic ionization rates
    - Chemical reaction rates
    - Comet and asteroid escape rates, resonance transition probabilities, collision probabilities

Transport Theory

- ***Transport in the solar system***
  - For objects of interest
    - e.g., Jupiter family comets, near-Earth asteroids, dust
  - **Identify phase space objects** governing transport
  - View N-body as multiple restricted 3-body problems
  - Look at stable and unstable manifold of periodic orbits associated with Lagrange points and mean motion resonances
  - Use these to **compute statistical quantities**
    - e.g., probability of resonance transition, escape rates

Transport Theory

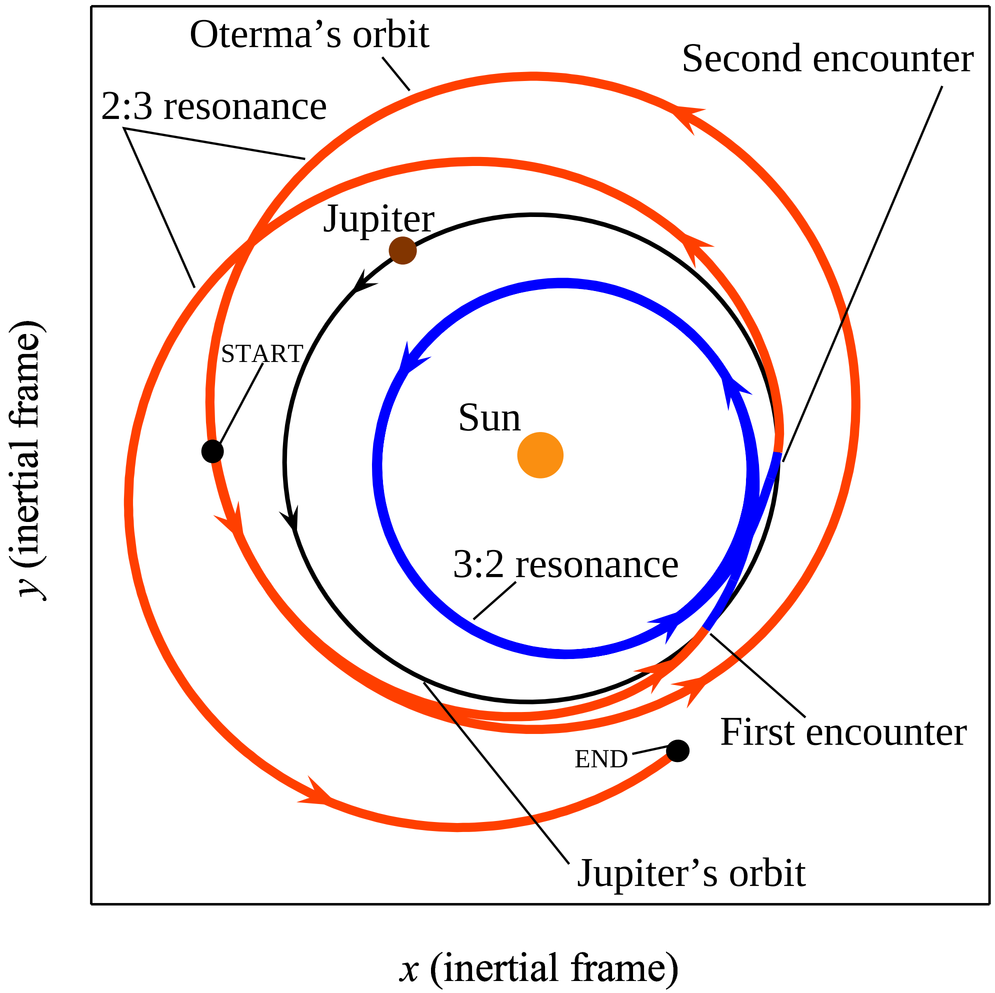
- of current astrophysical interest for understanding the transport of solar system material
  - eg, how ejecta gets from Mars to Earth
  - how likely is *Shoemaker-Levy 9*-type collision

Jupiter Family Comets

- ***Physical example of intermittency***
  - We consider the historical record of the comet Oterma from 1910 to 1980
    - first in an inertial frame
    - then in a rotating frame
    - a special case of pattern evocation
  - similar pictures exist for many other comets

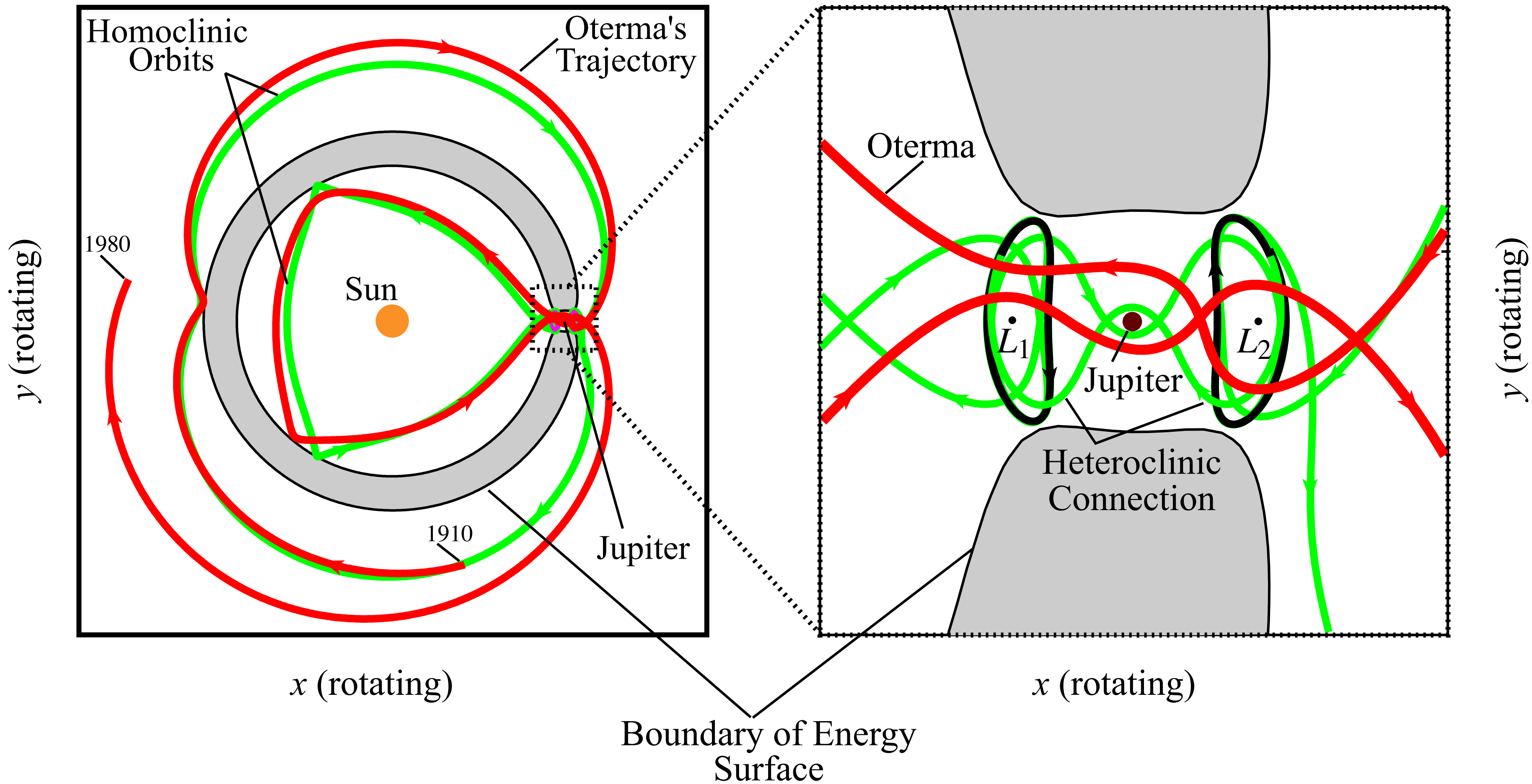
Jupiter Family Comets

- Rapid transition: outside to inside Jupiter's orbit.
  - Captured temporarily by Jupiter during transition.
  - Exterior (2:3 resonance) to interior (3:2 resonance).



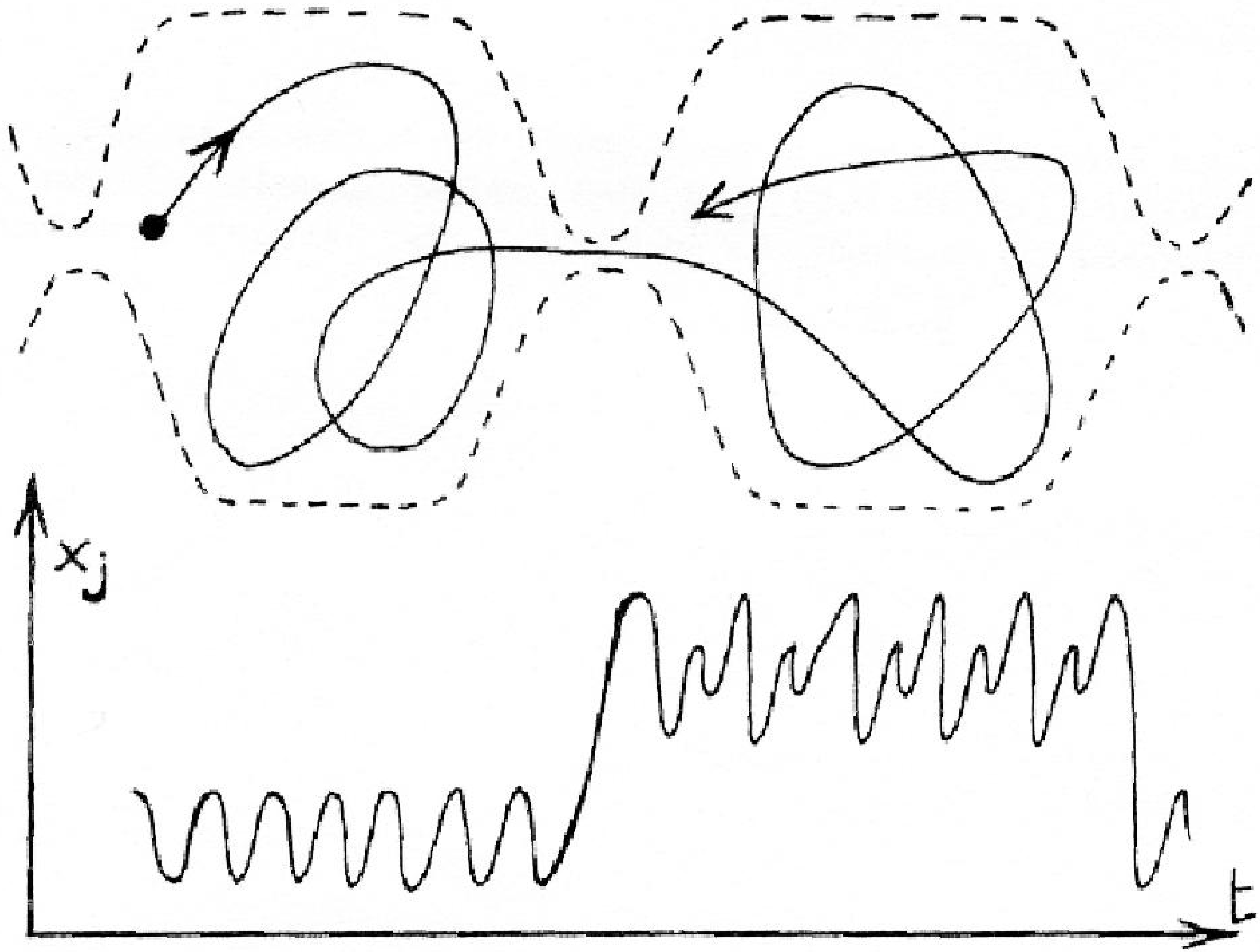
Viewed in Rotating Frame

- Oterma's orbit in rotating frame with some invariant manifolds of the 3-body problem superimposed.



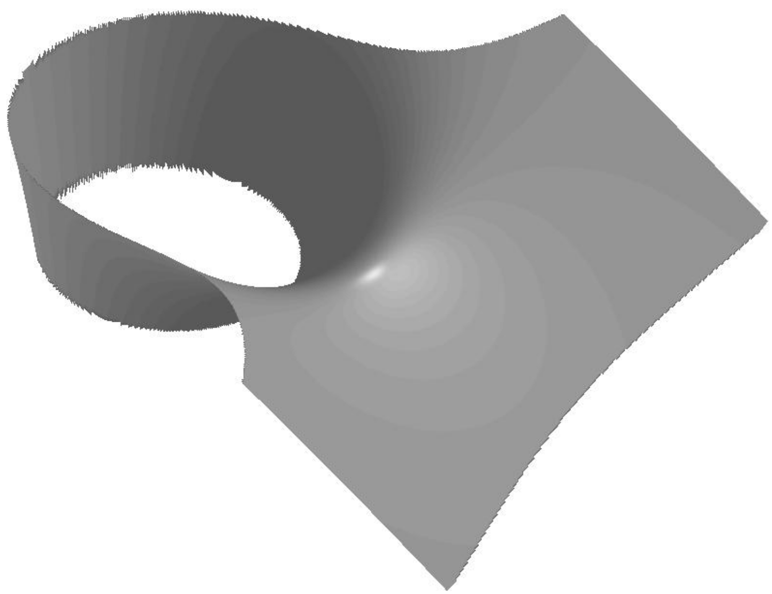
Partition the Phase Space

“Reactants” **“Products”**

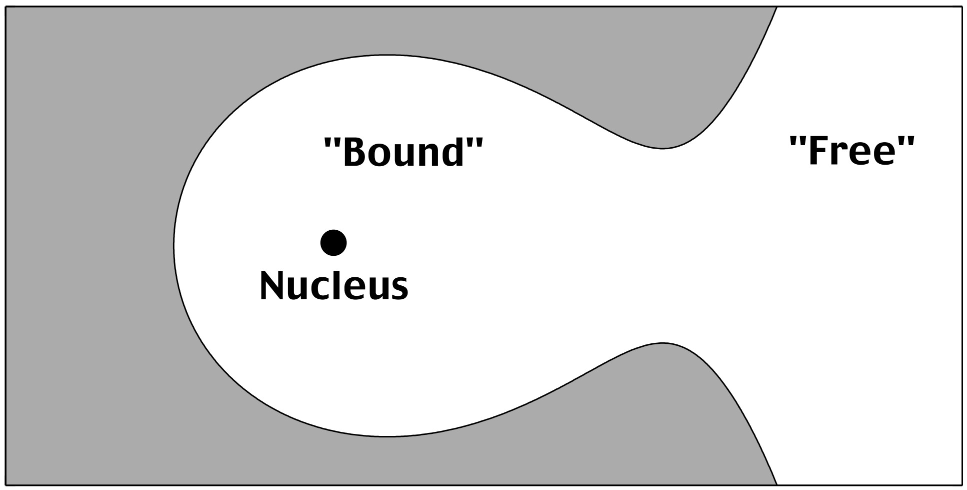


Partition the Phase Space

- ***Systems with potential barriers***
  - Electron near a nucleus with crossed electric and magnetic fields



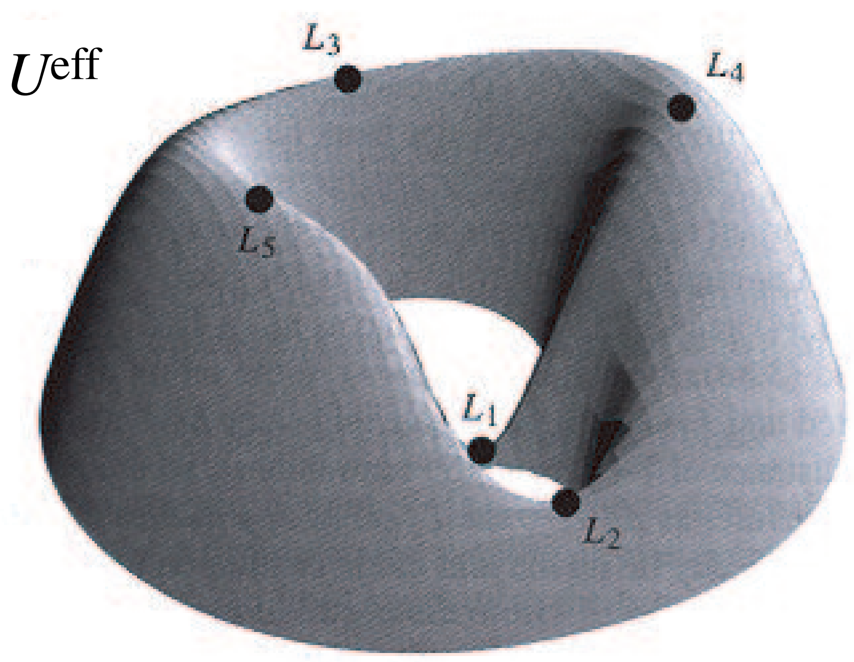
Potential



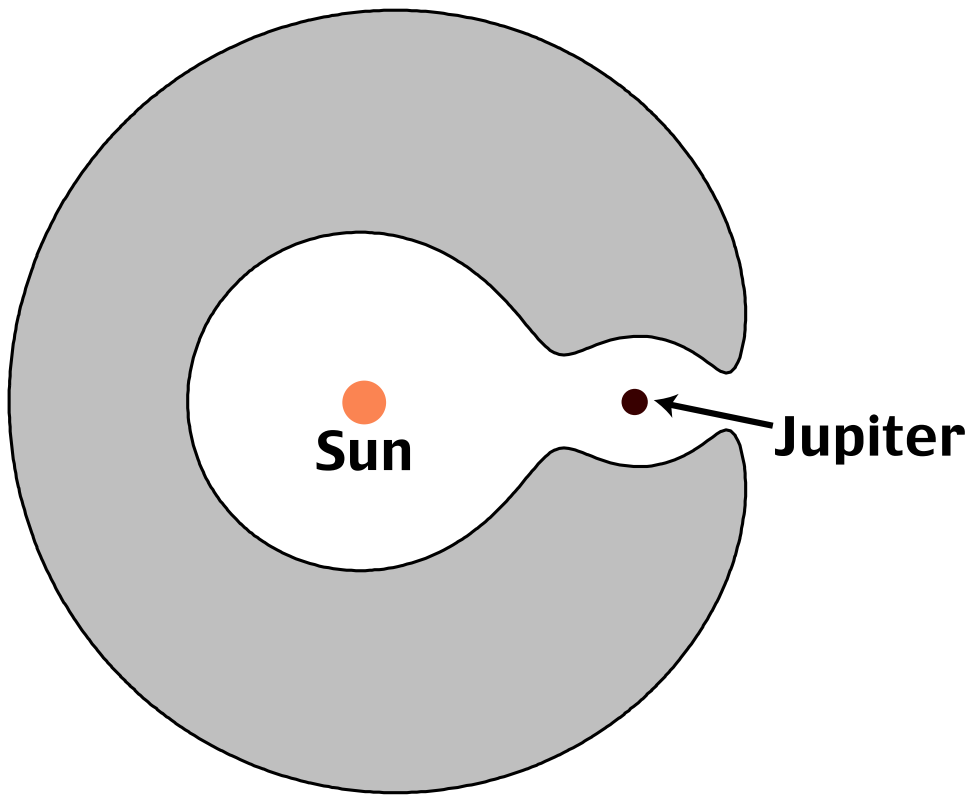
Configuration Space

Partition the Phase Space

- Comet near the Sun and Jupiter



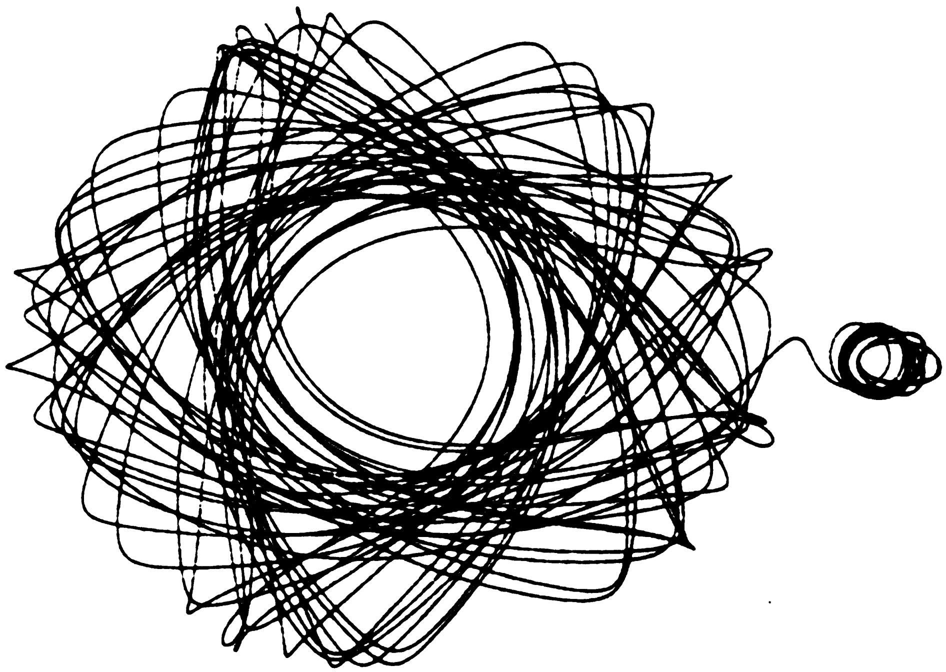
Potential



Configuration Space

Partition the Phase Space

- ***Partition is specific to problem***
  - We desire a way of describing dynamical boundaries that represent the "frontier" between qualitatively different types of behavior
- ***Example: motion of a comet***
  - motion around the Sun
  - motion around Jupiter



Statement of Problem

- Suppose we study the motion on a manifold $\mathcal{M}$
- Suppose $\mathcal{M}$ is partitioned into disjoint regions

  $$R_i, i = 1, \ldots, N_R,$$

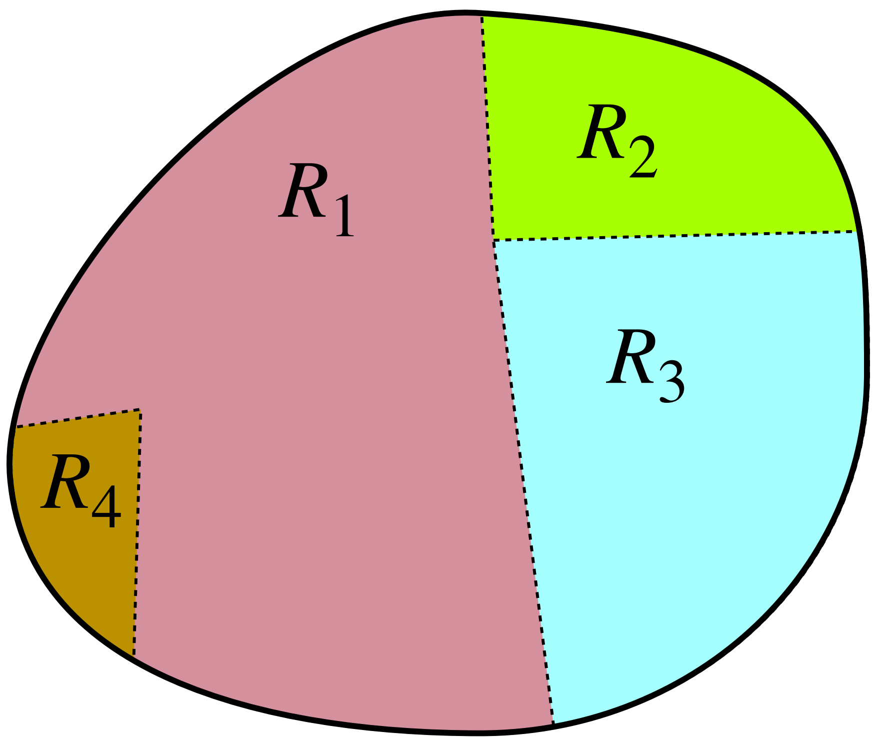
  such that

  $$\mathcal{M} = \bigcup_{i=1}^{N_R} R_i.$$

- To keep track of the initial condition of a point, we say that *initially* (at $t = 0$) region R_i is uniformly covered with species S_i.
- Thus, species type of a point indicates the region in which it was located initially.

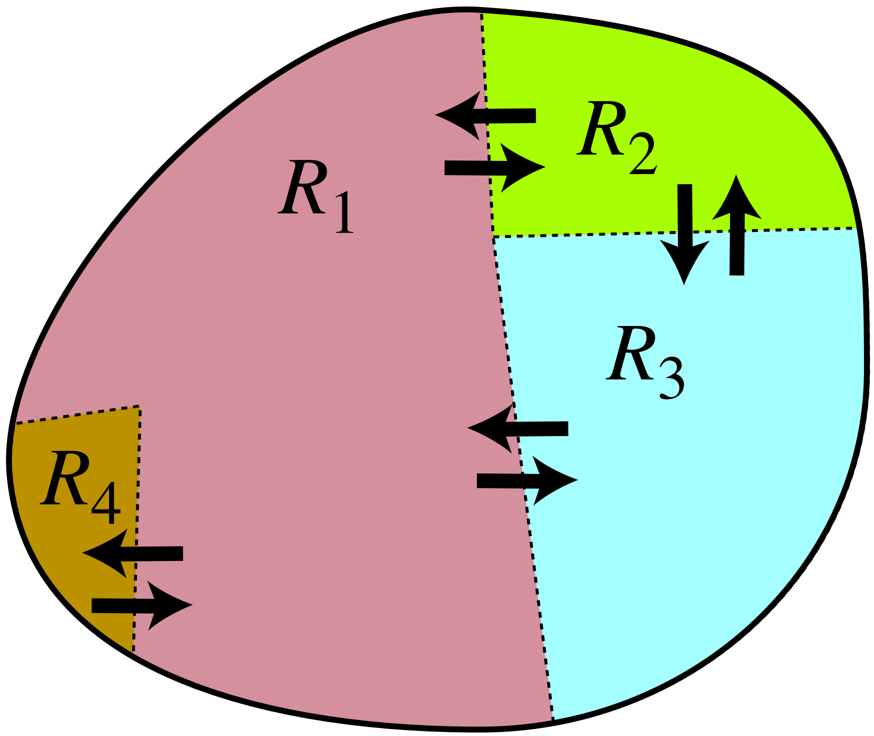
Statement of Problem

- Statement of the transport problem: Describe the distribution of species $S_i, i = 1, \ldots, N_R$, throughout the regions $R_j, j = 1, \ldots, N_R$, for any time $t > 0$.



Statement of Problem

- Some quantities we would like to compute are:
  $T_{i,j}(t) =$ the total amount of species S_i contained in region R_j at time t
  $F_{i,j}(t) = \frac{dT_{i,j}}{dt}(t) =$ the flux of species S_i into region R_j at time t



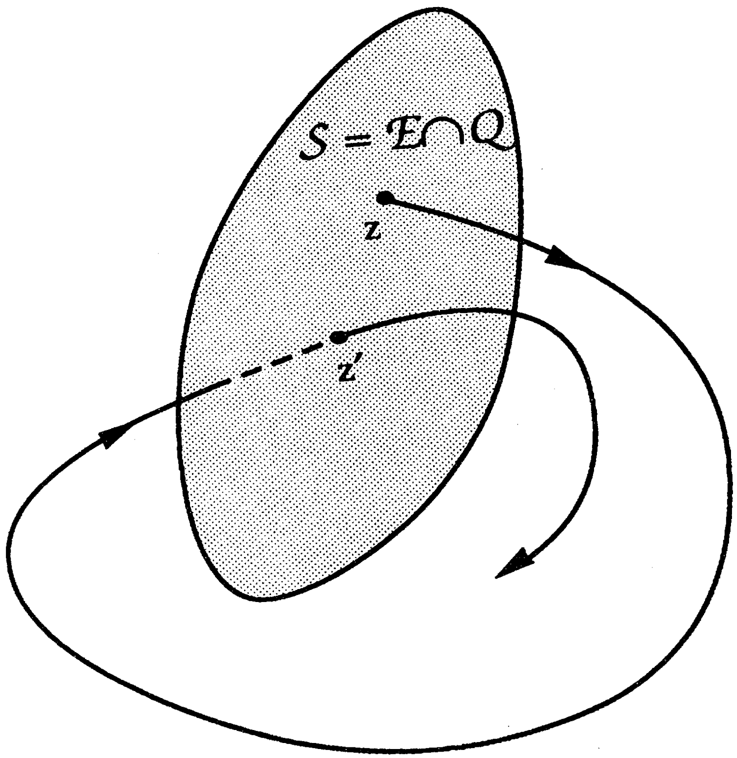
Hamiltonian Systems

- ***Time-independent Hamiltonian*** $H(q,p)$
  - N degrees of freedom
  - Motion constrained to a $(2N-1)$-dimensional energy surface $\mathcal{M}_E$ corresponding to a value $H(q,p) = E = \text{constant}$
  - Symplectic area is conserved along the flow

$$\oint_{\mathcal{L}} p \cdot dq = \int_{\mathcal{A}} dp \wedge dq = \text{constant}$$

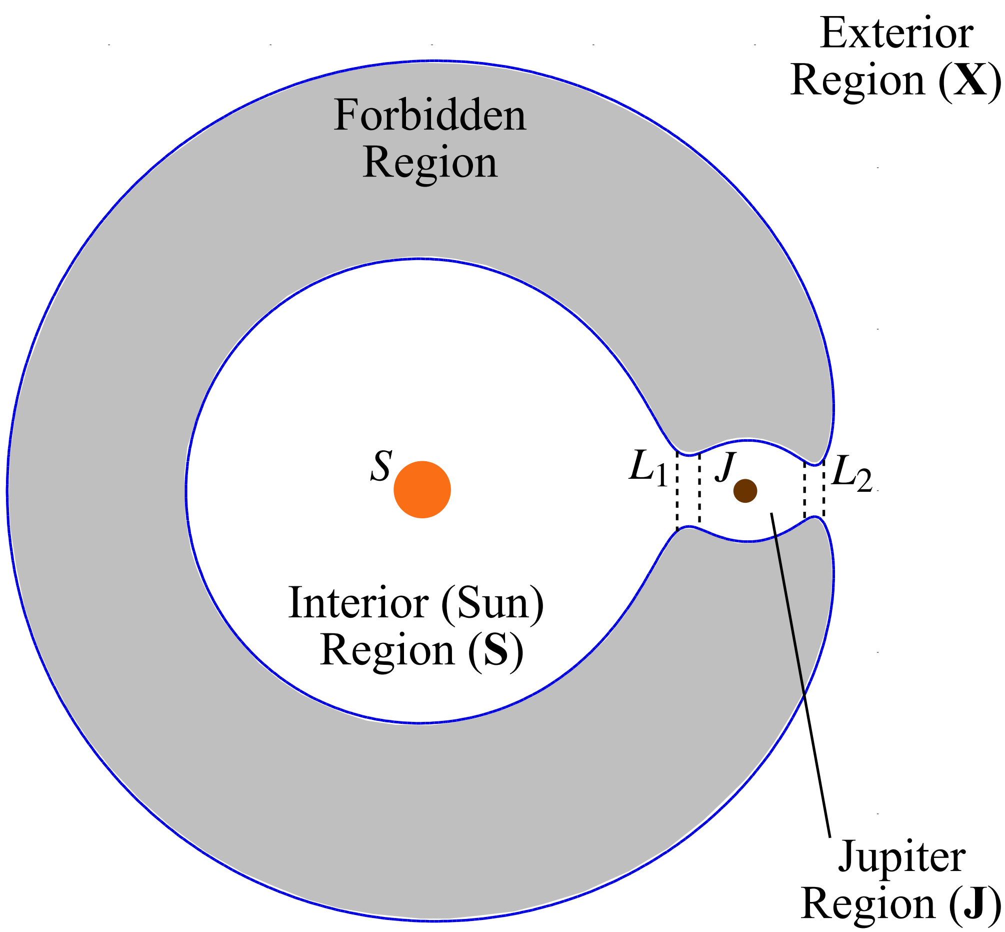
Poincaré Section

- Suppose there is another $(2N-1)$-dimensional surface $\mathcal{Q}$ that is transverse (i.e., nowhere parallel) to the flow in some local region.
- The Poincaré section $\mathcal{S}$ is the $(2N-2)$-dimensional intersection of $\mathcal{M}_E$ with $\mathcal{Q}$.



Example for N=2

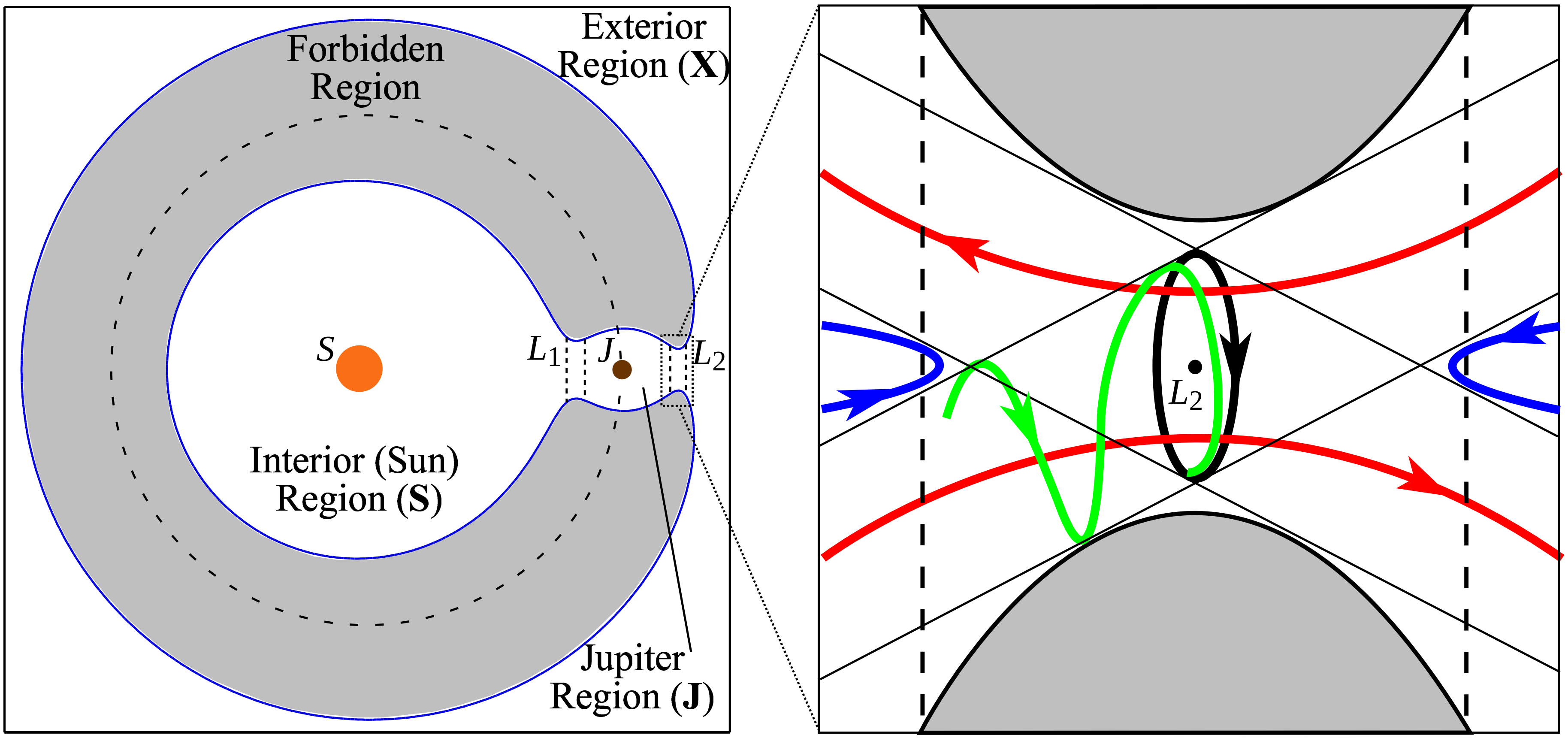
- ***Restricted 3-body problem (planar)***
  - Partition the energy surface: **S, J, X regions**



Position Space Projection

Equilibrium Region

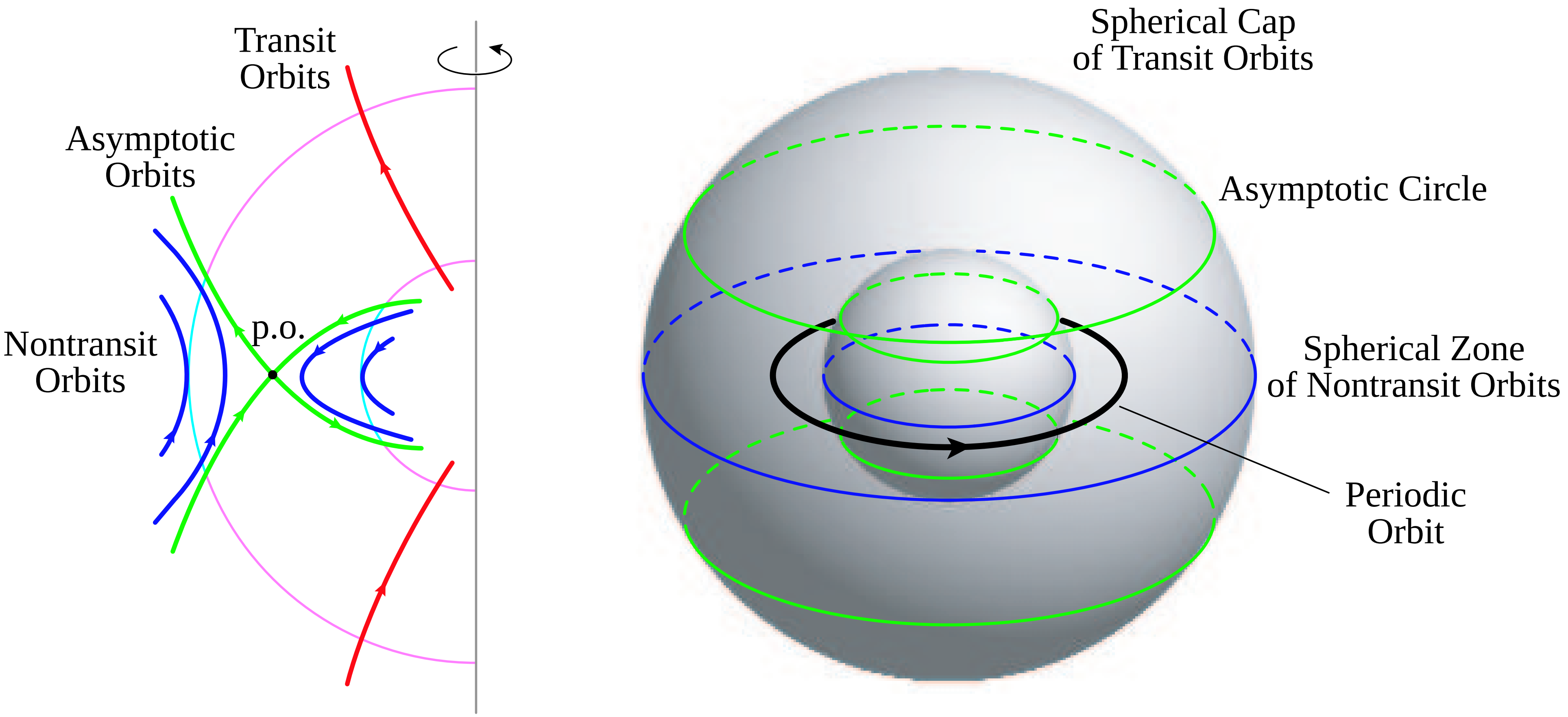
- ***Look at motion near the potential barrier, i.e. the equilibrium region***



Position Space Projection

Local Dynamics

- For fixed energy, the equilibrium region $\simeq S^2 \times \mathbb{R}$.
  - Stable/unstable manifolds of periodic orbit define mappings between bounding spheres on either side of the barrier



Cross-section of Equilibrium Region

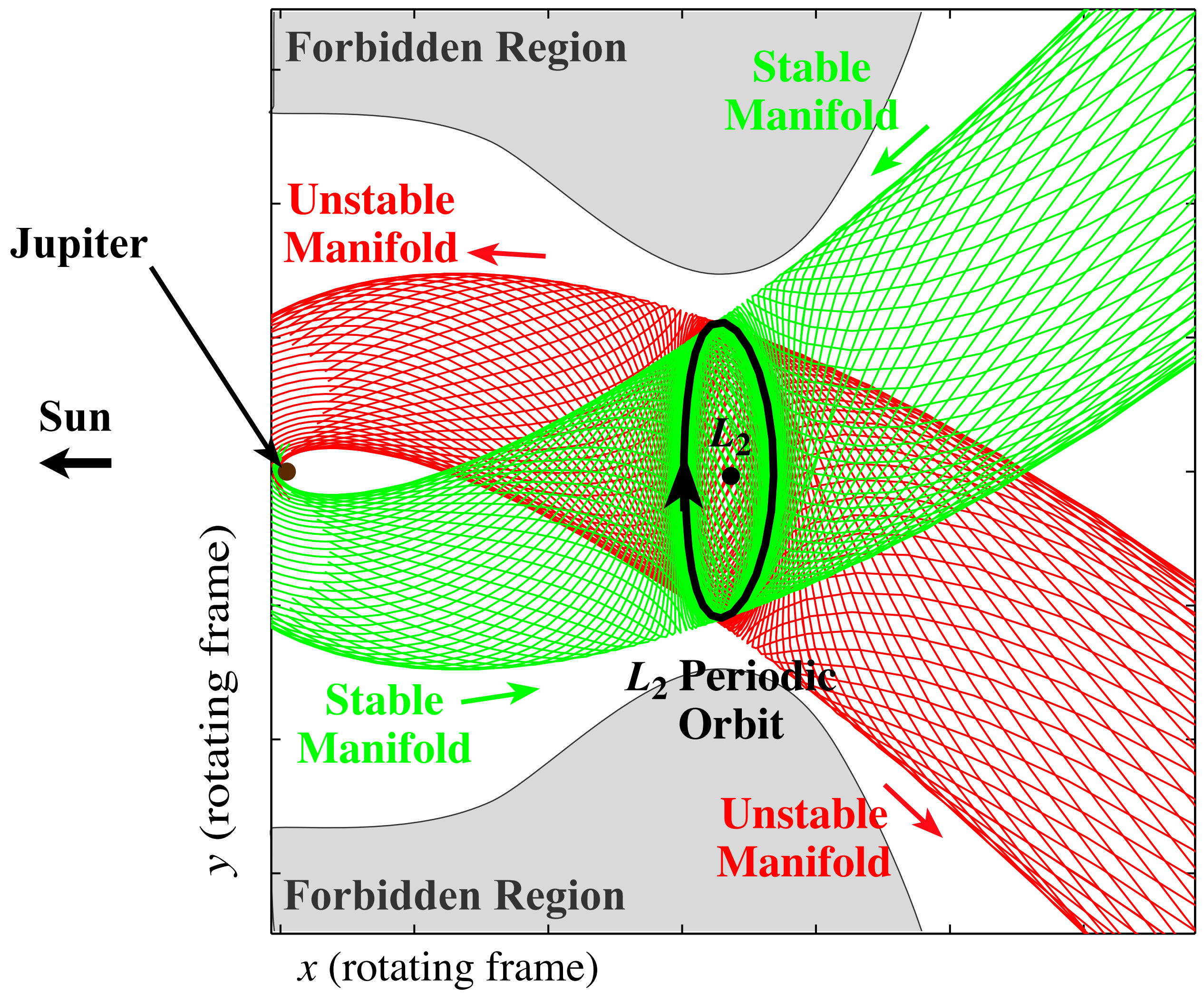
Equilibrium Region

Transition State Theory

- This is related to the **transition state theory** of the chemical literature.
- Wiggins, Wiesenfeld, Jaffé, and Uzer [2001] extend transition state theory to higher dimensional systems.
- Interesting connection between chemical and celestial dynamics!

Tubes in the 3-Body Problem

- **Stable** and **unstable** **manifold tubes**
  - Control transport through the potential barrier.



Flux between Regions

- ***Tubes of transit orbits are the relevant objects to study***
  - Tubes determine the **flux** between regions $F_{i,j}(t)$.
  - Net flux is zero for volume-preserving motion, so we consider the **one-way flux**
  - Example: $F_{J,S}(t) =$ volume of trajectories that escape from the Jupiter region into the Sun region per unit time.

Transition Rates

- ***Fluxes give rates and probabilities***
  - Jaffé, Ross, Lo, Marsden, Farrelly, and Uzer [2002] computed the rate of escape of asteroids temporarily captured by Mars.
  - RRKM-like statistical approach
    - similar to chemical dynamics, see Truhlar [1996]
  - Consider an asteroid (or other body) in orbit around Mars (perhaps impact ejecta) at a 3-body energy such that it can escape toward the Sun.
  - Interested in rate of escape of such bodies at a fixed energy, i.e. $F_{M,S}(t)$

Transition Rates

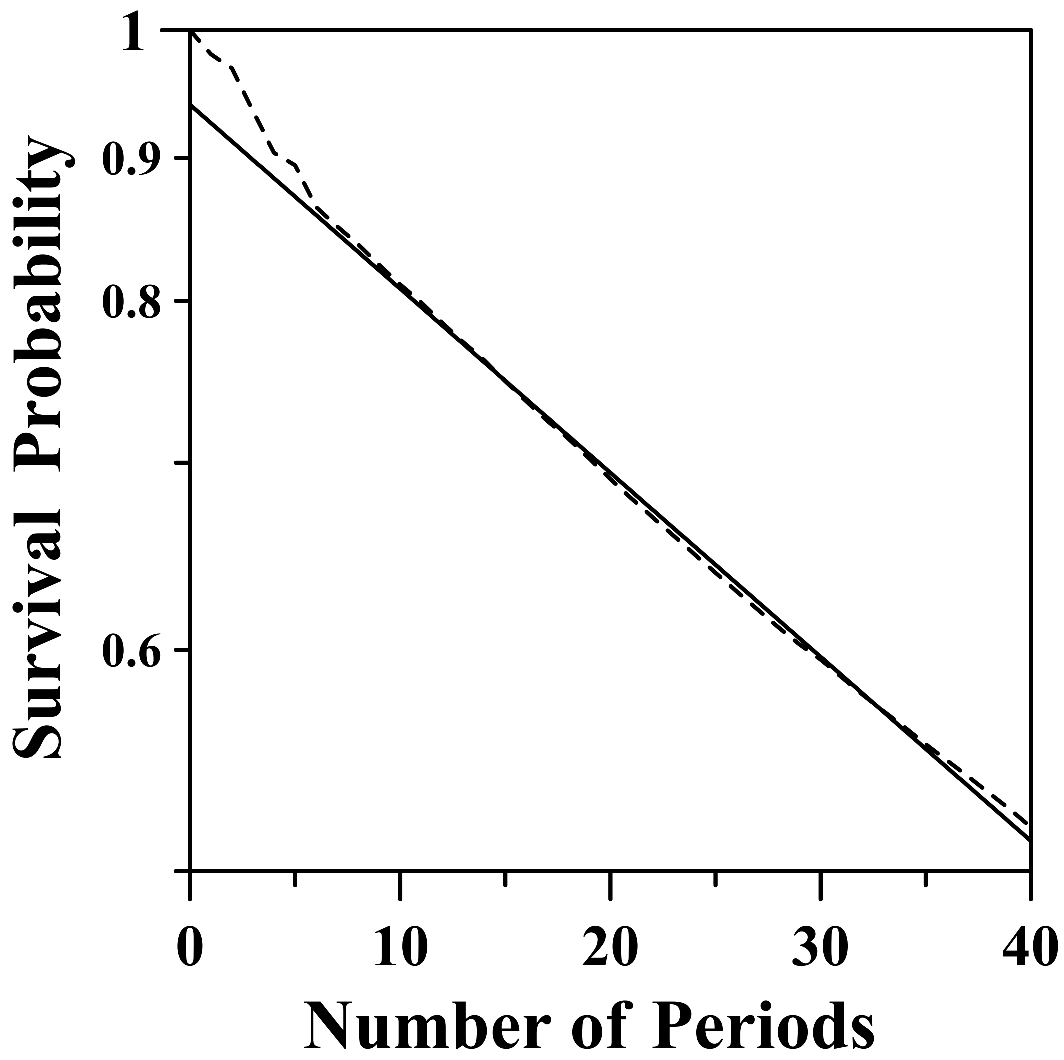
- RRKM assumption: all asteroids in the Mars region at fixed energy are **equally likely to escape**. Then

$$\text{Escape rate} = \frac{\text{flux across potential barrier}}{\text{Mars region phase space volume}}$$

- Compare with Monte Carlo simulations of 107,000 particles
  - randomly selected initial conditions at constant energy

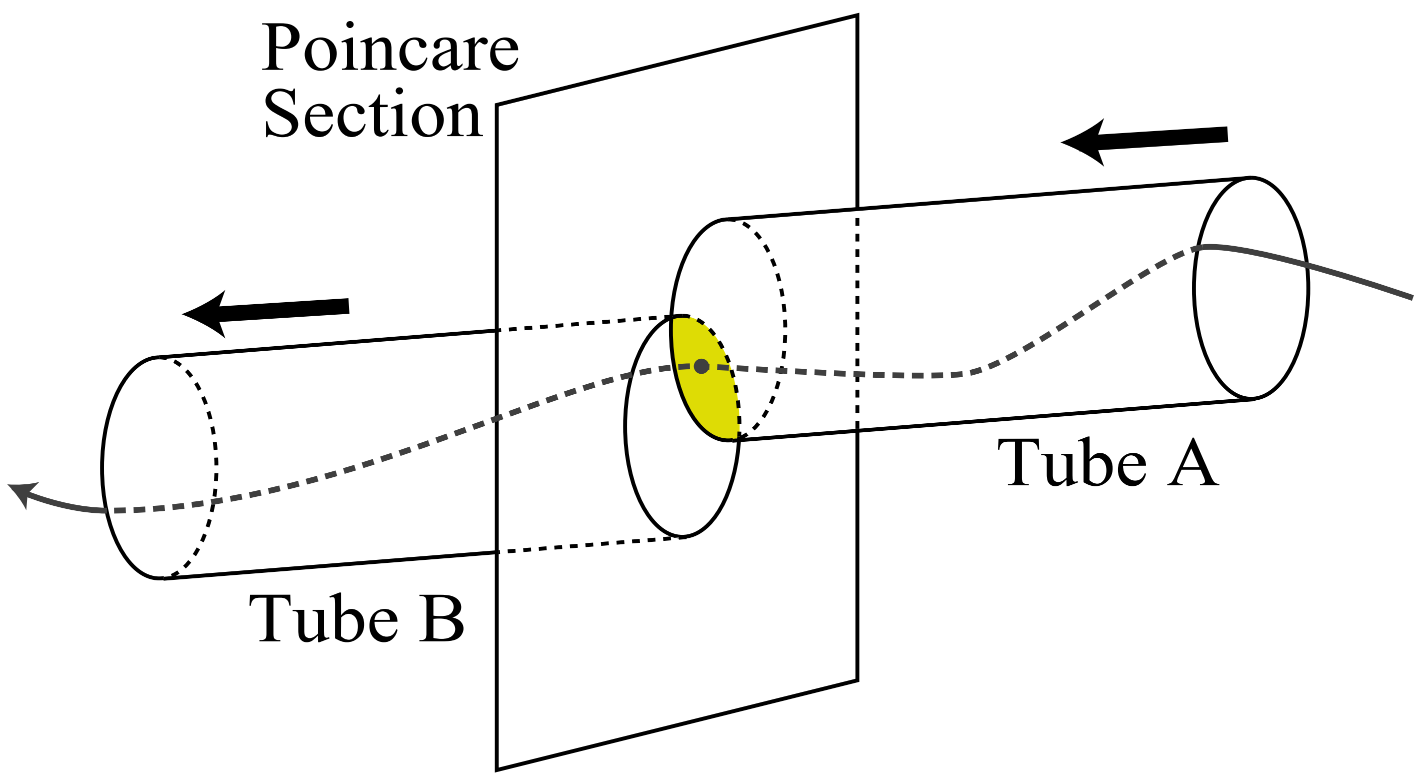
Transition Rates

- Theory and numerical simulations agree well.
  - Monte Carlo simulation (dashed) and theory (solid)



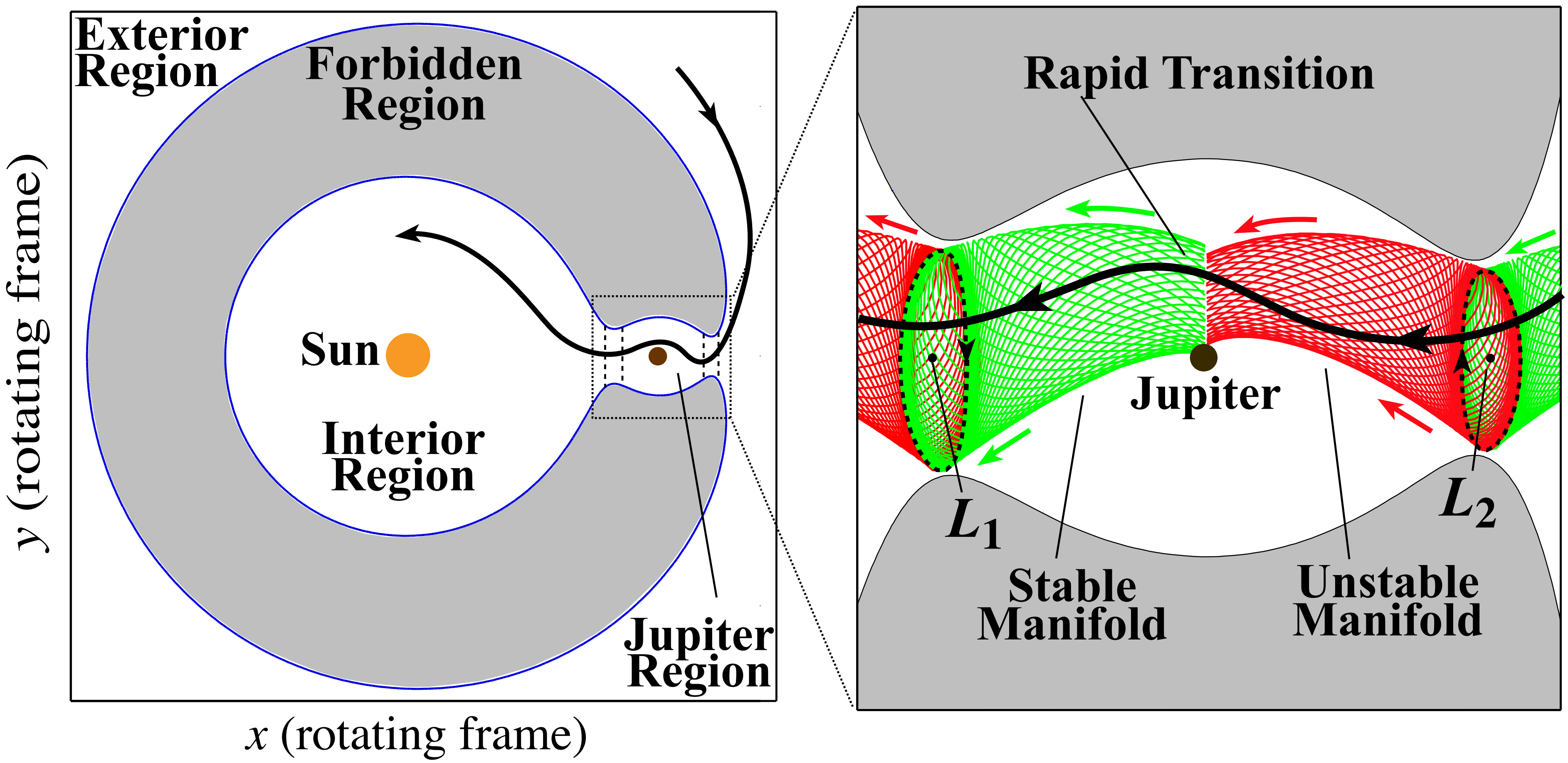
Transition Rates

- *More exotic transport between regions*
  - Look at the intersections between the interior of stable and unstable tubes on the same energy surface.
  - Could be from different potential barriers.



Transition Rates

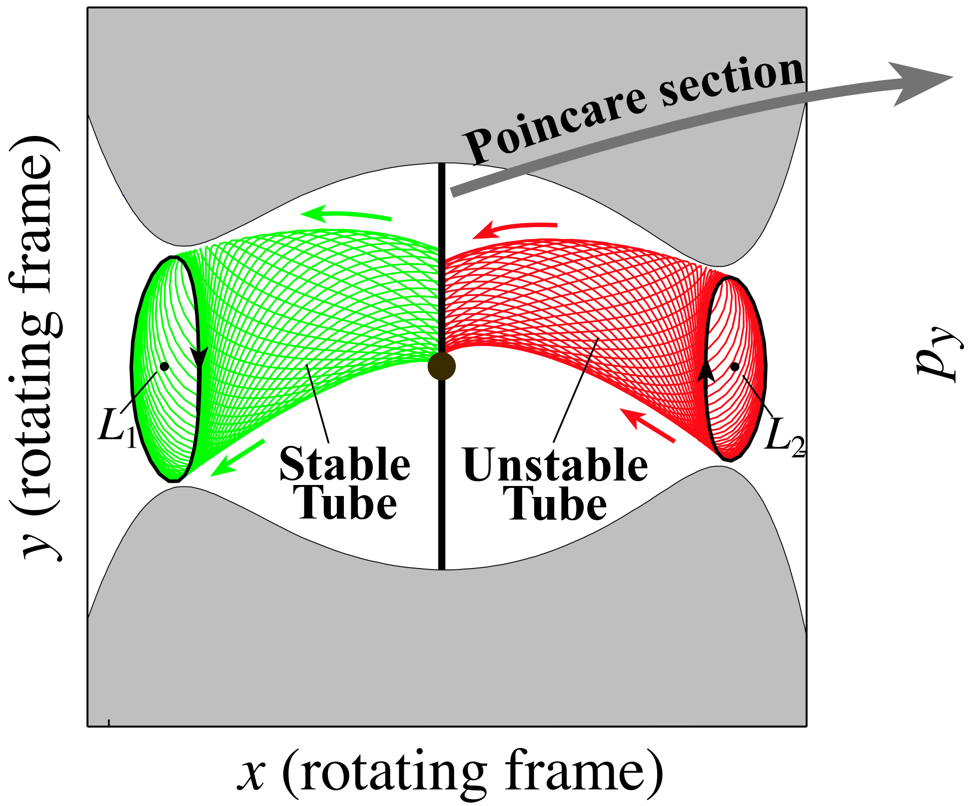
- Example: Comet transport between outside and inside of Jupiter (i.e., Oterma-like transitions)



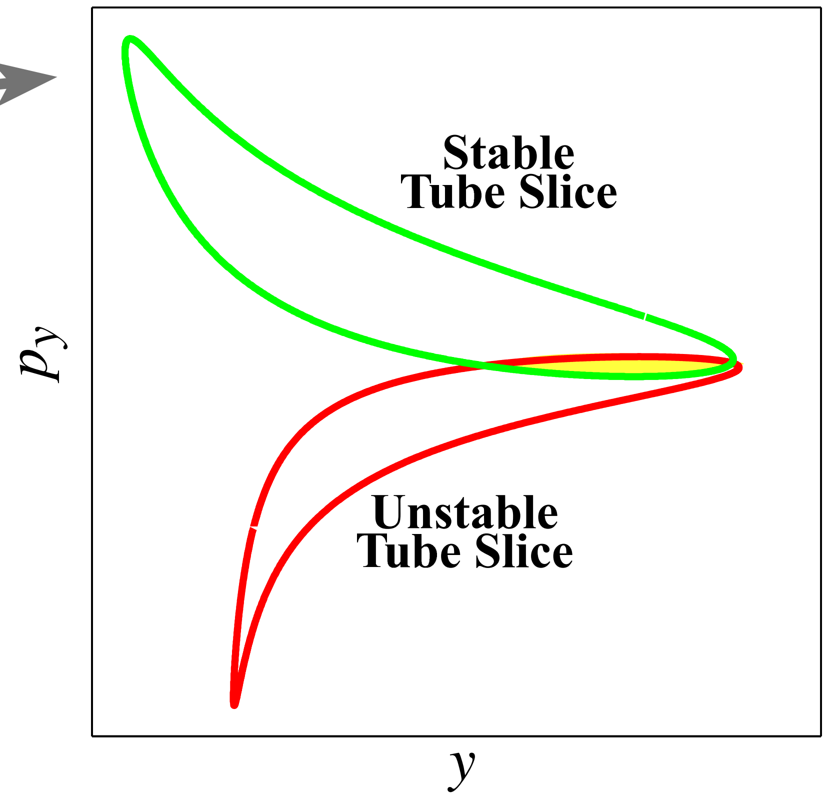
(a)

Transition Rates

- Look at Poincaré section intersected by both tubes.
- Choosing surface $\{x = \text{constant}; p_x < 0\}$, we look at the canonical plane (y, p_y).



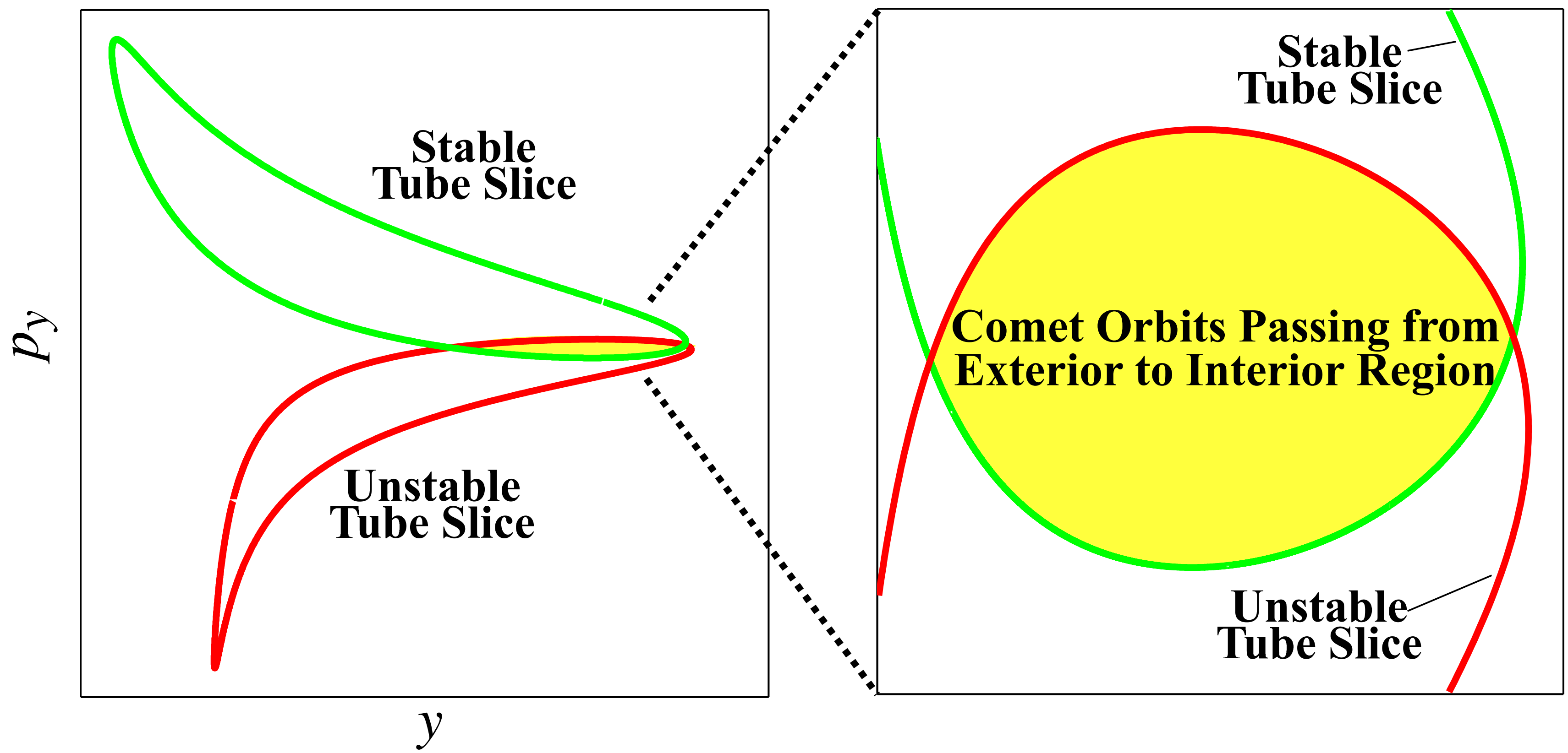
Position Space



Canonical Plane (y, p_y)

Transition Rates

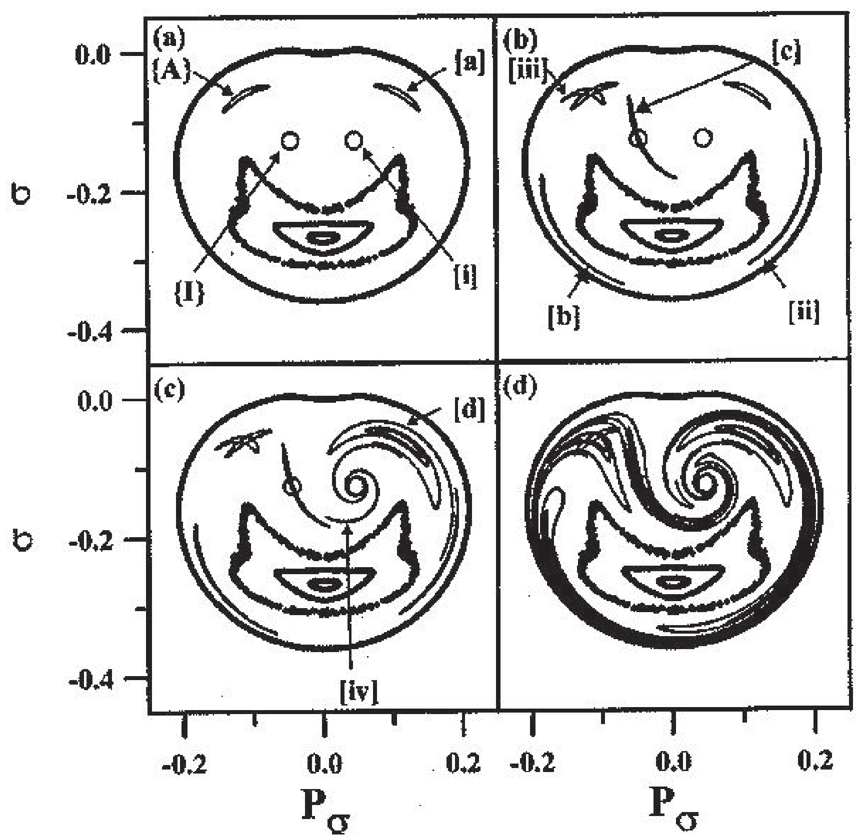
- Relative canonical area gives relative flux of orbits.
- Under RRKM assumptions, can compute probability of transition from one region to another.



Canonical Plane (y, p_y)

Mixing

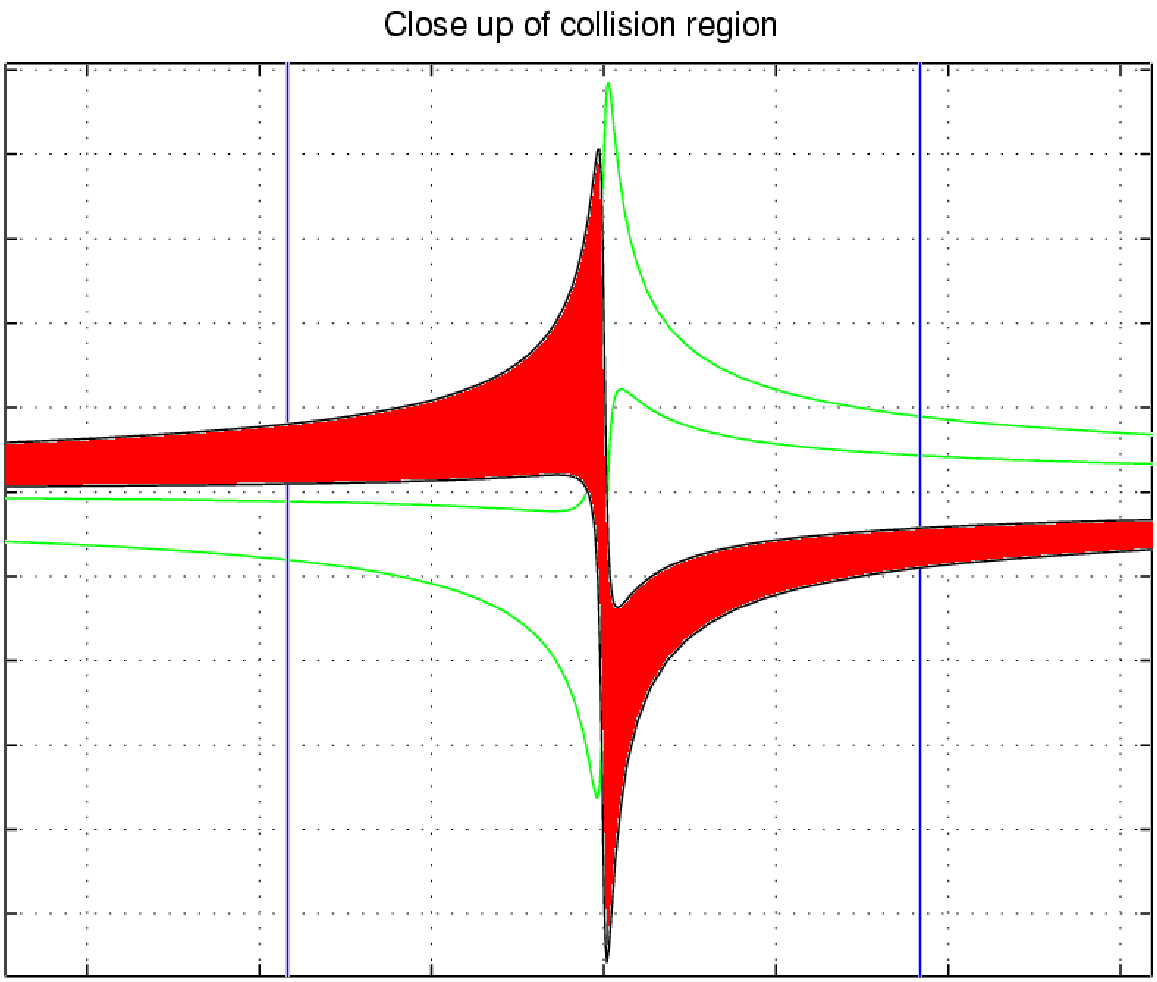
- By keeping track of the intersections of the tubes, one can describe the mixing of different regions ($T_{i,j}(t)$).
  - It can get complicated!
  - Example: atomic transition rates (Jaffé, Farrelly, Uzer [1999])



Collision

Collision probabilities

- computed from tube intersection with planet on Poincaré section

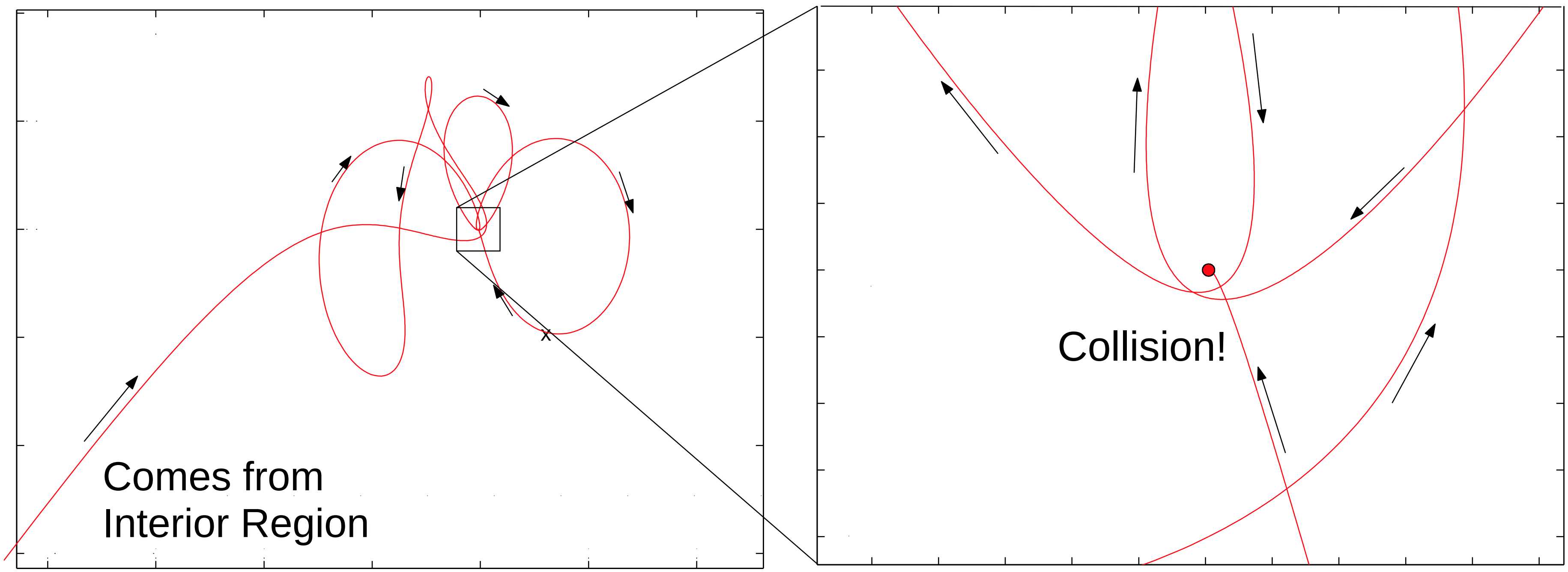


← Diameter of planet →

Collision

- Statistical approach to low velocity impact probabilities
  - eg, *Shoemaker-Levy 9* and Earth crossers

Example Collision Trajectory



Conclusion and Future Work

- ***Transport in the solar system***
  - View solar system as many restricted 3-body problems
  - Planar restricted 3-body problem
    - Stable and unstable manifold tubes of periodic orbits can be used to compute statistical quantities of interest
    - Asteroid escape problem: first application of RRKM-like statistical approach to celestial mechanics
    - Theory and numerical simulation agree well

Conclusion and Future Work

Future Work

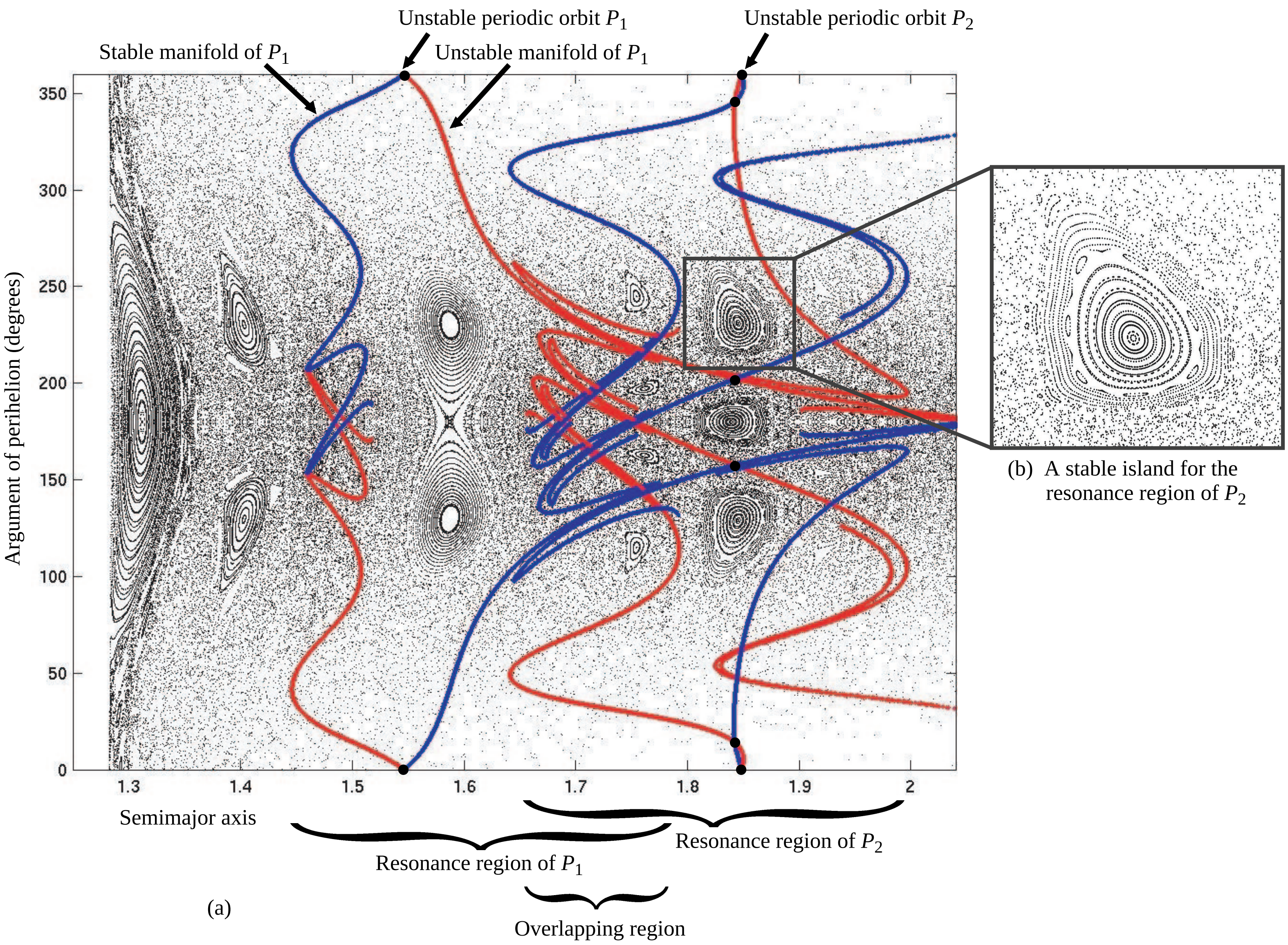
- Extend planar results to spatial problem — theory makes computation in high dimensions much easier
- Transport between mean motion resonances
  - Slow migration between resonances leading to temporary capture by or close encounter with a planet.
- Transport between planets
  - e.g., comets switching “control” from Saturn to Jupiter
- Consider drag-perturbed case
  - e.g., interplanetary dust particles

Transport between MMRs

- Transport rates between mean motion resonances (MMRs) can be computed via a **lobe dynamics** approach (see Wiggins [1992]).
- Several statistical quantities of interest can be computed as a function of planetary mass and particle energy.
  - average trapping time in a $p:q$ MMR
  - flux entering $p:q$ MMR from $p\prime:q\prime$ MMR

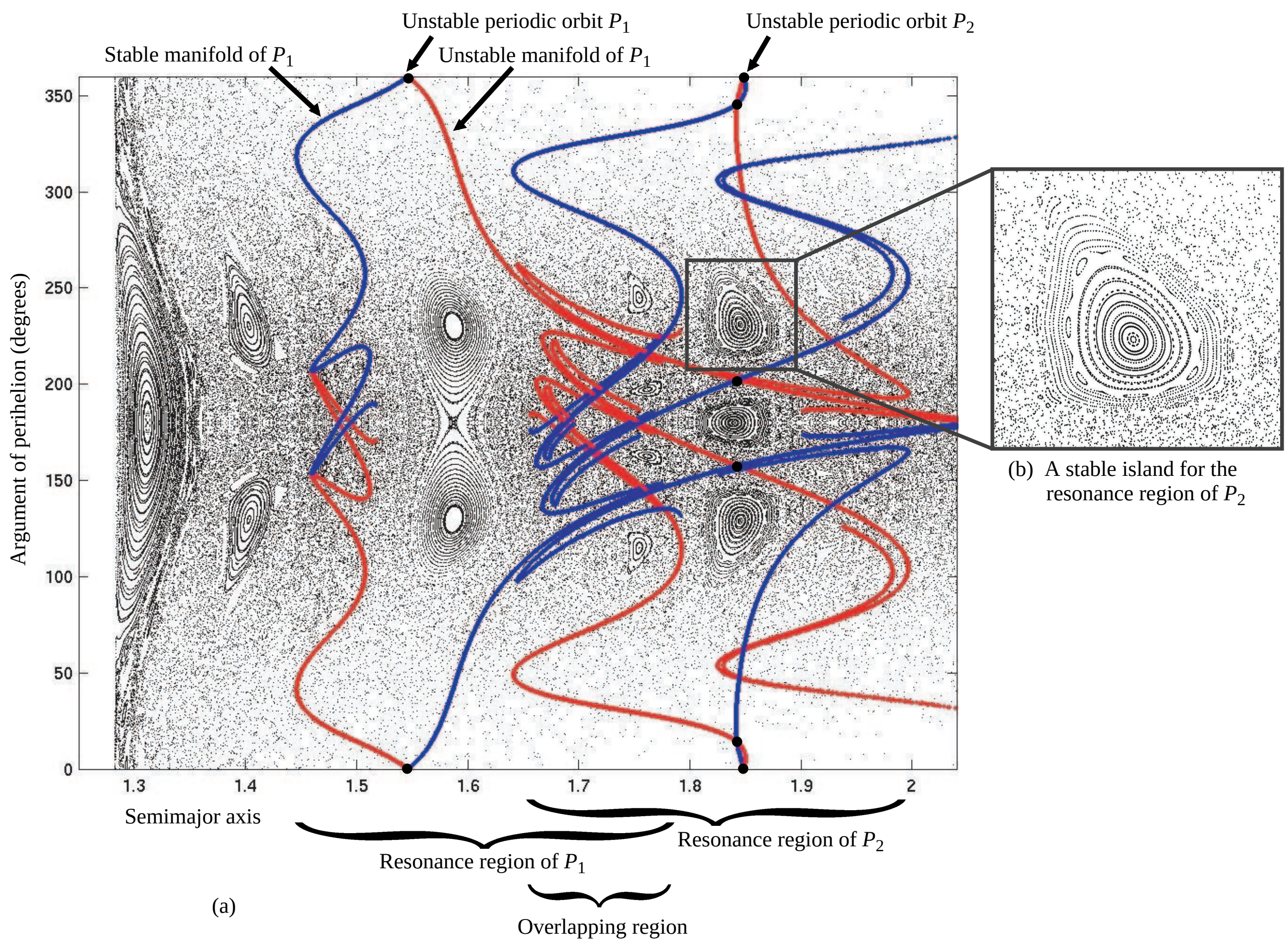
Transport between MMRs

We can compute the resonance regions



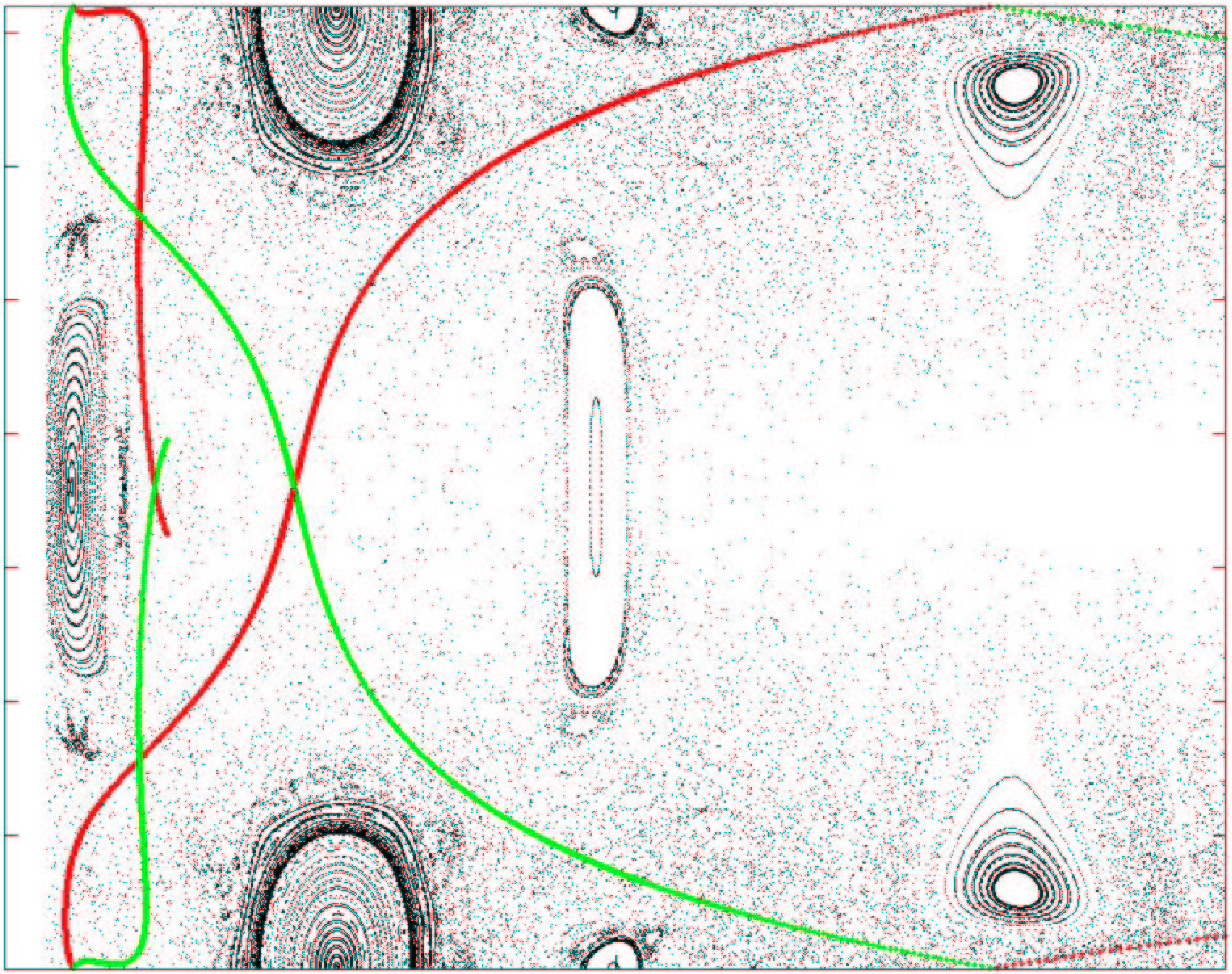
Transport between MMRs

A direct transition from a $p : q$ to a $p\prime : q\prime$ MMR is possible only if the exit lobe of a $p : q$ turnstile overlaps with the entry lobe of a $p\prime : q\prime$ turnstile.



MMRs and Close Encounters

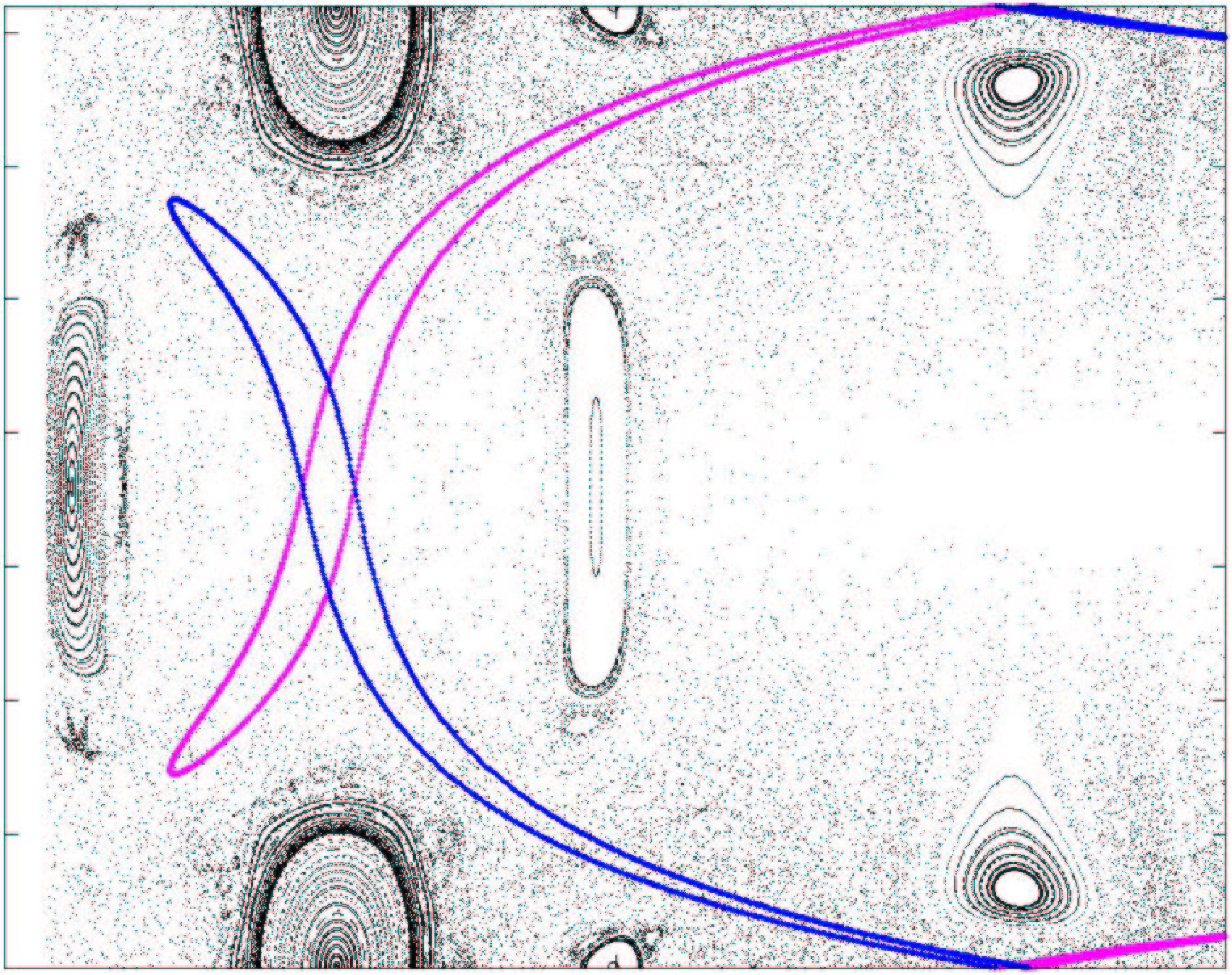
Poincaré section: plot resonance regions



2:3 exterior MMR with Jupiter

MMRs and Close Encounters

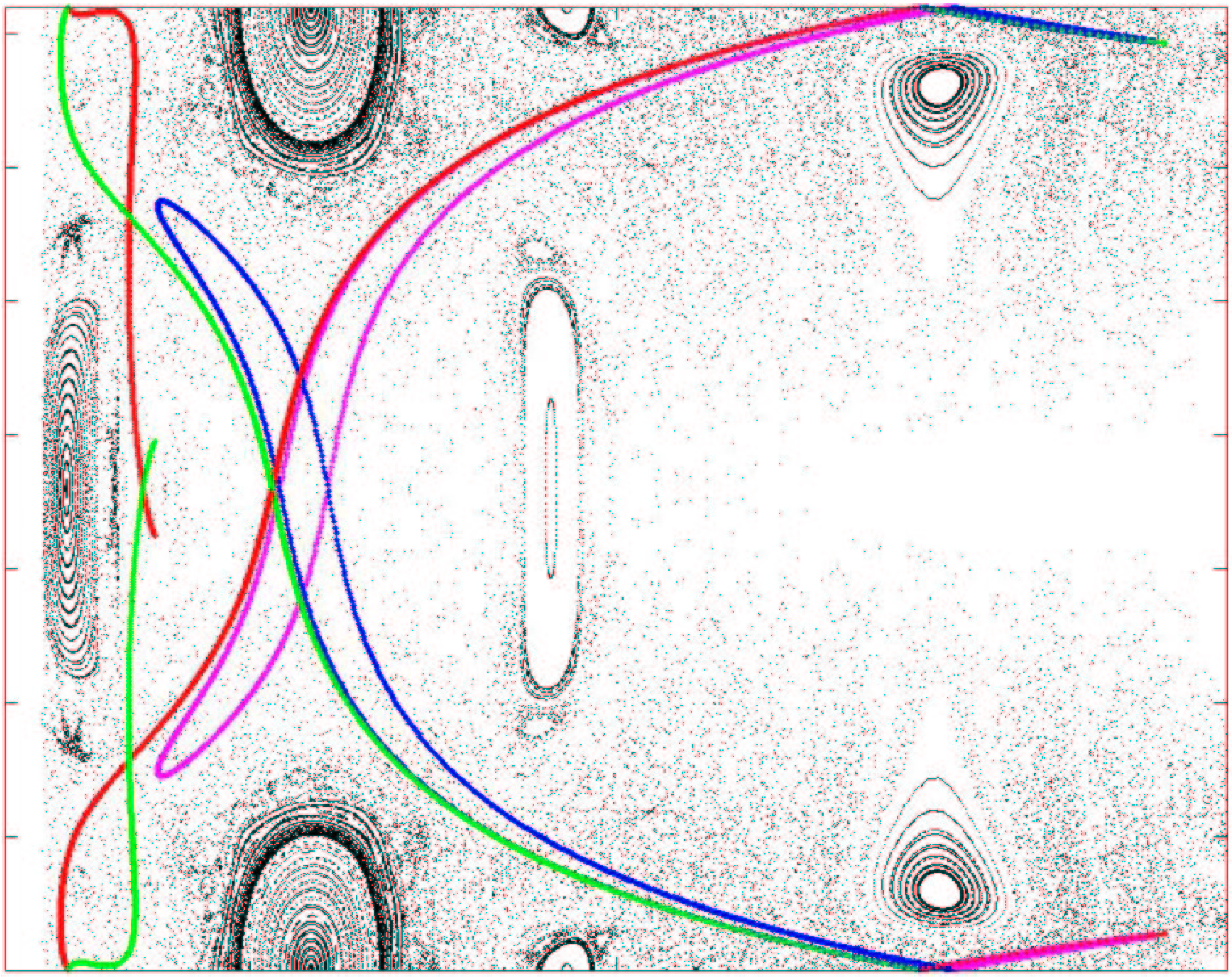
Same section: tube cross-sections are closed curves



Particles inside curves move toward or away from Jupiter

MMRs and Close Encounters

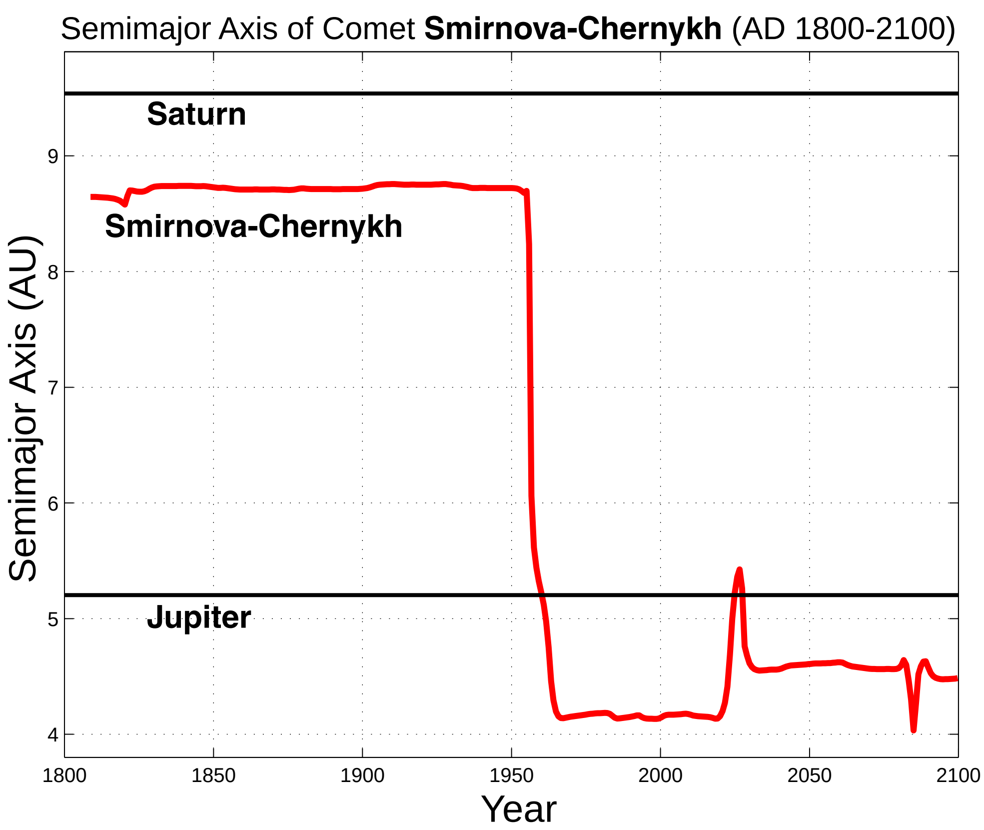
Regions of overlap lead to/from close encounters

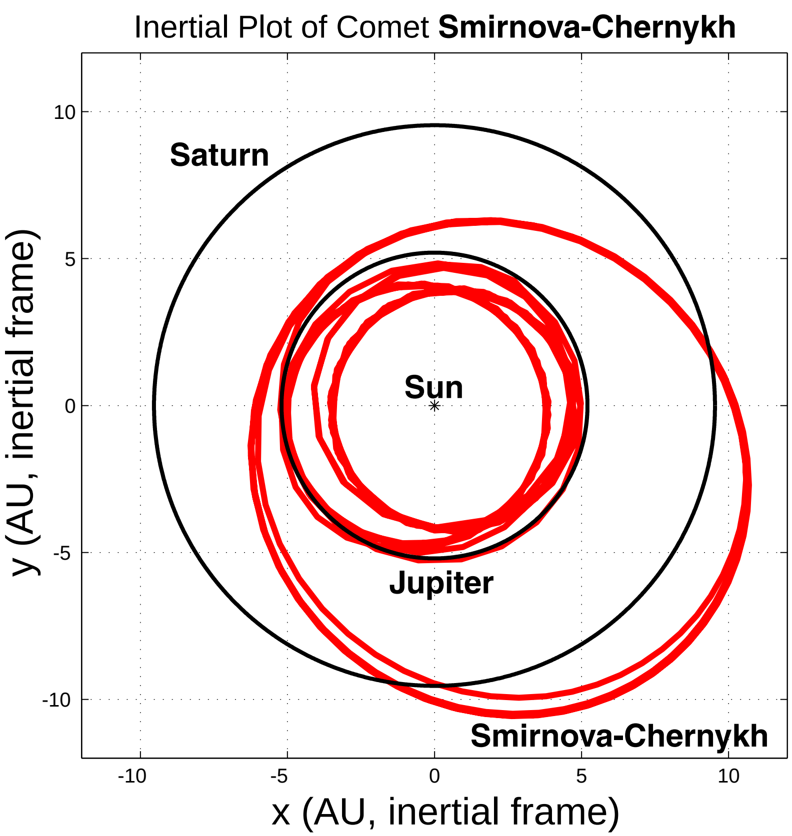


Regions of overlap occur

Transport between Planets

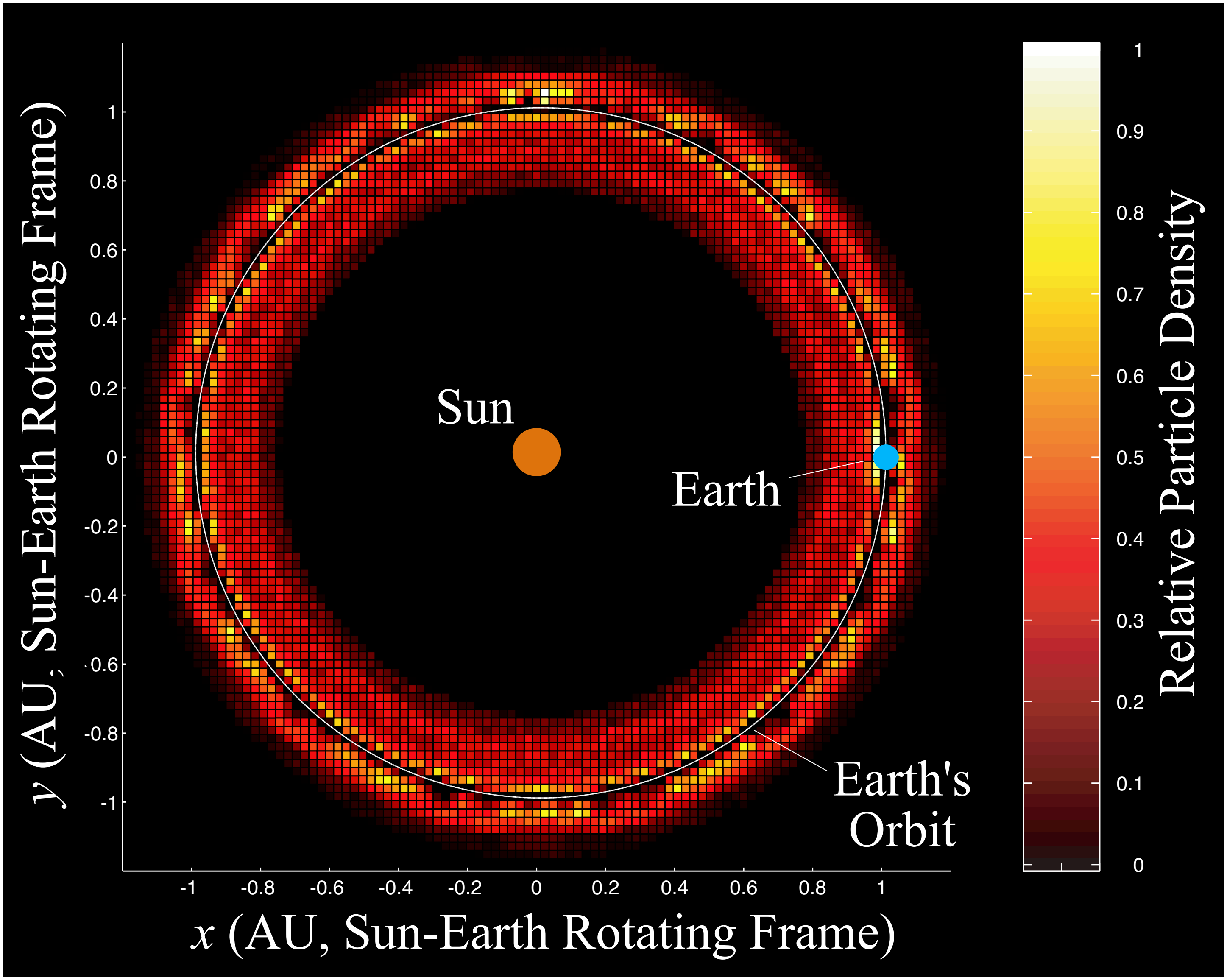
Comets transfer between the giant planets
eg, jumping between “tubes” of Saturn and Jupiter





Circumstellar Dust Clouds

Drag-perturbed case important for planet-finding



References

- Jaffé, C., S.D. Ross, M.W. Lo, J. Marsden, D. Farrelly, and T. Uzer [2002] ***Statistical theory of asteroid escape rates***. *Physical Review Letters*, to appear.
- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2001] ***Resonance and capture of Jupiter comets.*** *Celestial Mechanics and Dynamical Astronomy* 81(1-2), 27–38.
- Gómez, G., W.S. Koon, M.W. Lo, J.E. Marsden, J. Masdemont and S.D. Ross [2001] ***Invariant manifolds, the spatial three-body problem and space mission design.*** *AAS/AIAA Astrodynamics Specialist Conference.*
- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000] ***Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics.*** *Chaos* 10(2), 427–469.

For papers, movies, etc., visit the website:
http://www.cds.caltech.edu/~shane

The End