CALTECH
Control & Dynamical Systems

Transport in Hamiltonian Systems With Two or More Degrees of Freedom

Shane D. Ross

Control and Dynamical Systems, Caltech

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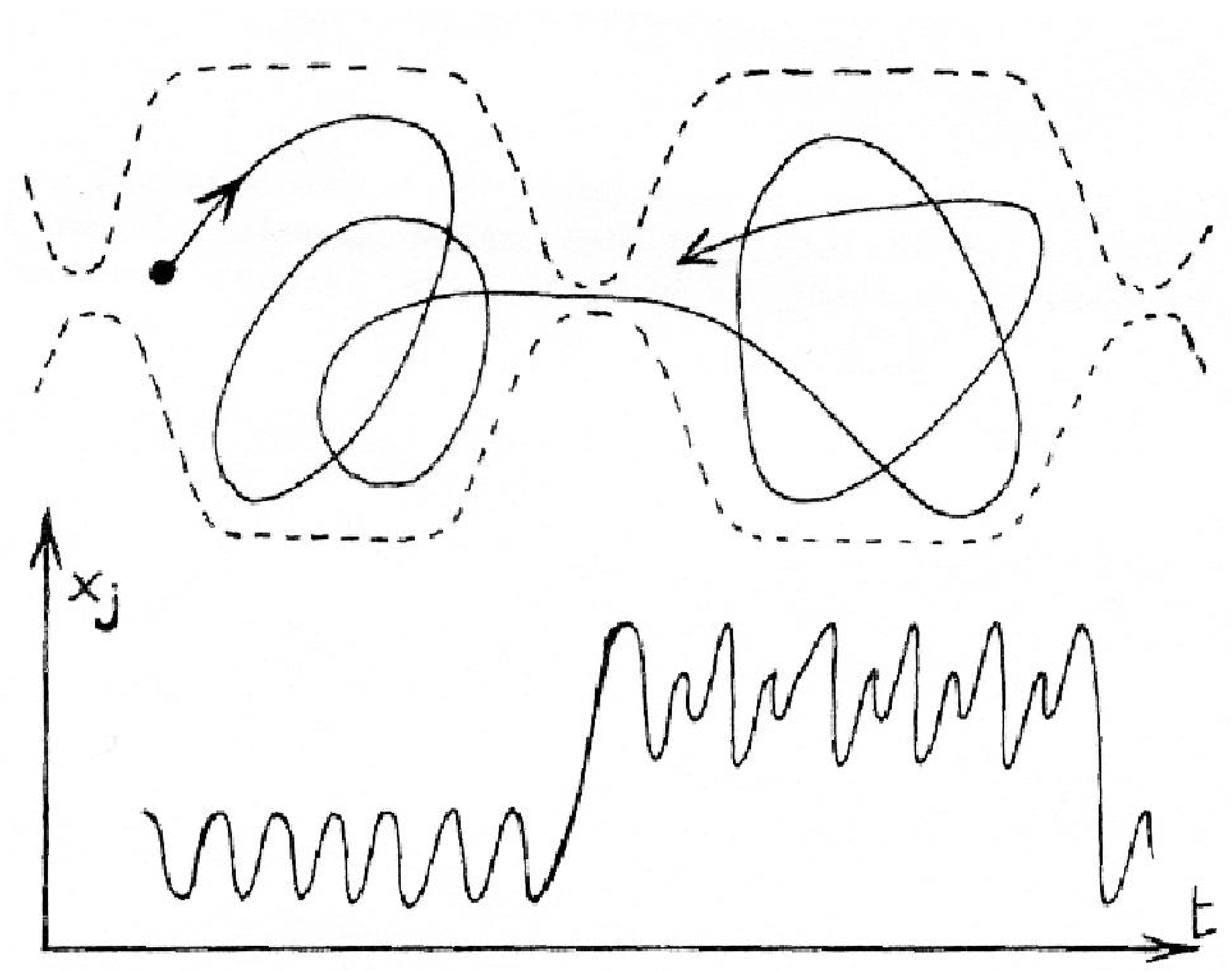
Overview

- ***Transport theory***
  - Time-independent Hamiltonian systems
  - with 2 degrees of freedom
  - with 3 (or N) degrees of freedom
    - Example: restricted three-body problem
  - Some notes on computation

Chaotic Dynamics



Transport Theory

- *Chaotic dynamics*
  $\Longrightarrow$ *statistical methods*

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  - Motion of ensembles of trajectories in phase space
  - Asks: How long to move from one region to another?
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    - Chemical reaction rates

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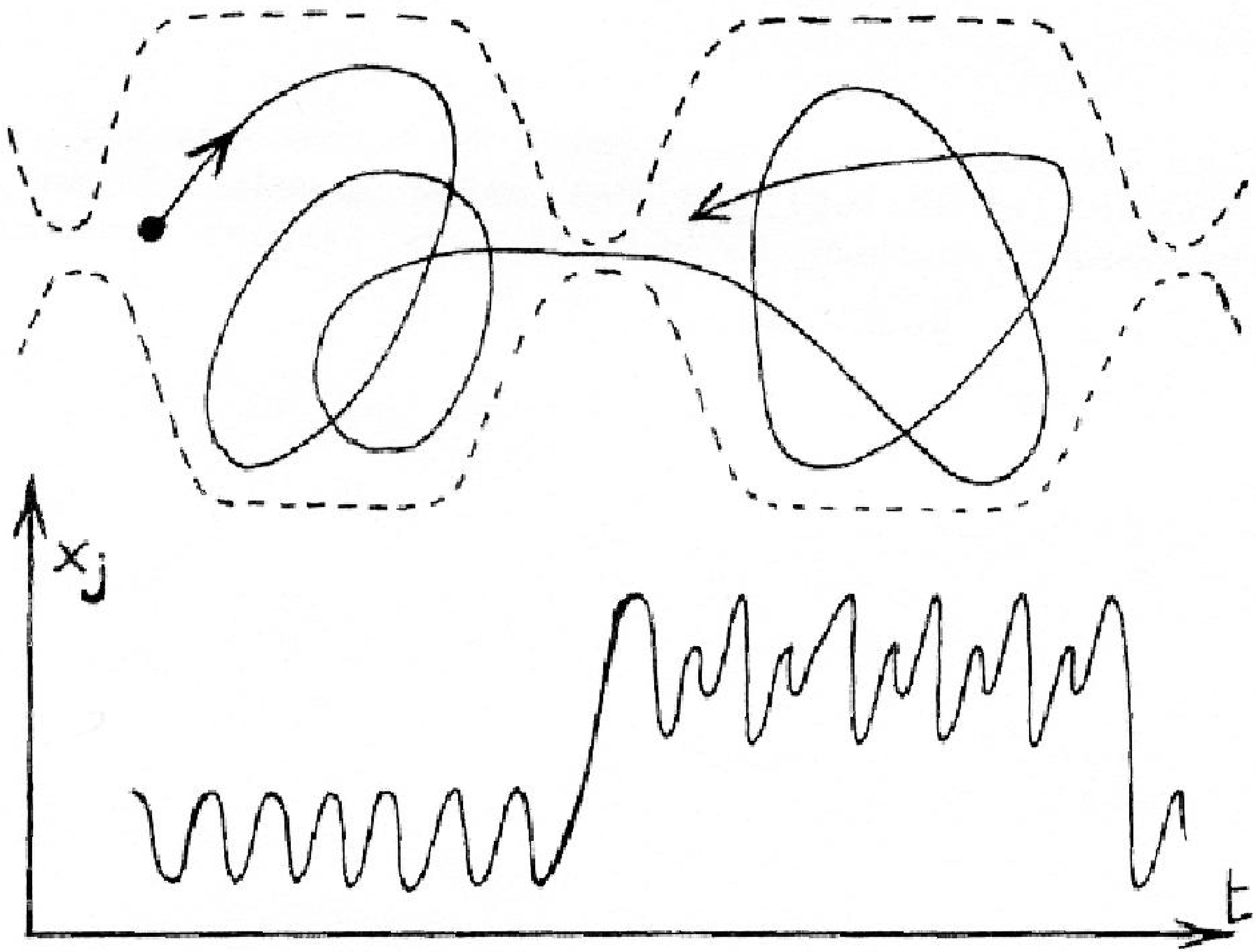
  - Determine transition probabilities, escape rates
  - Applications:
    - Atomic ionization rates
    - Chemical reaction rates
    - Comet transition rates
    - Asteroid collision probabilities

Partition the Phase Space

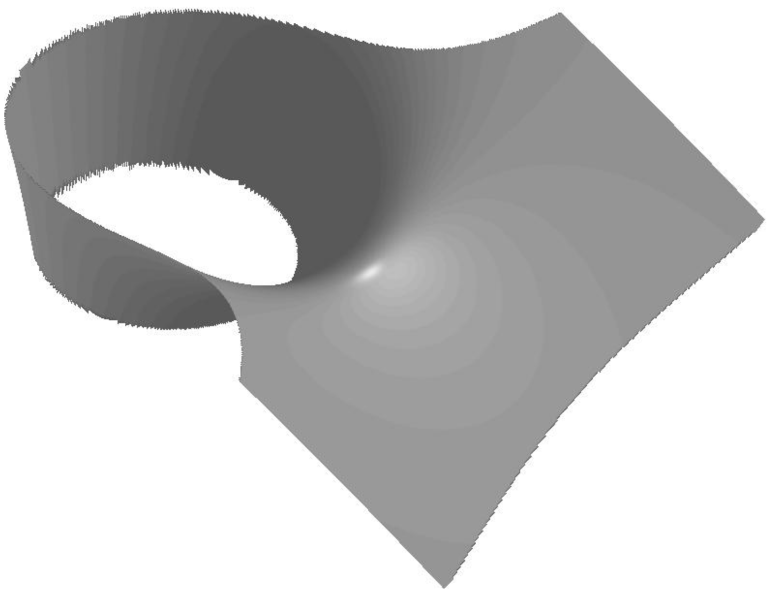
“Reactants” “Products”



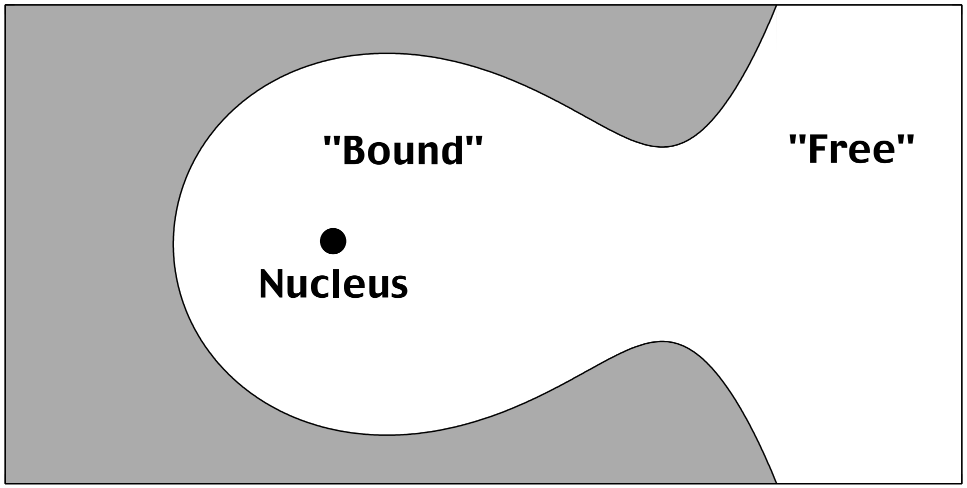
Partition the Phase Space

Systems with potential barriers

- Electron near a nucleus



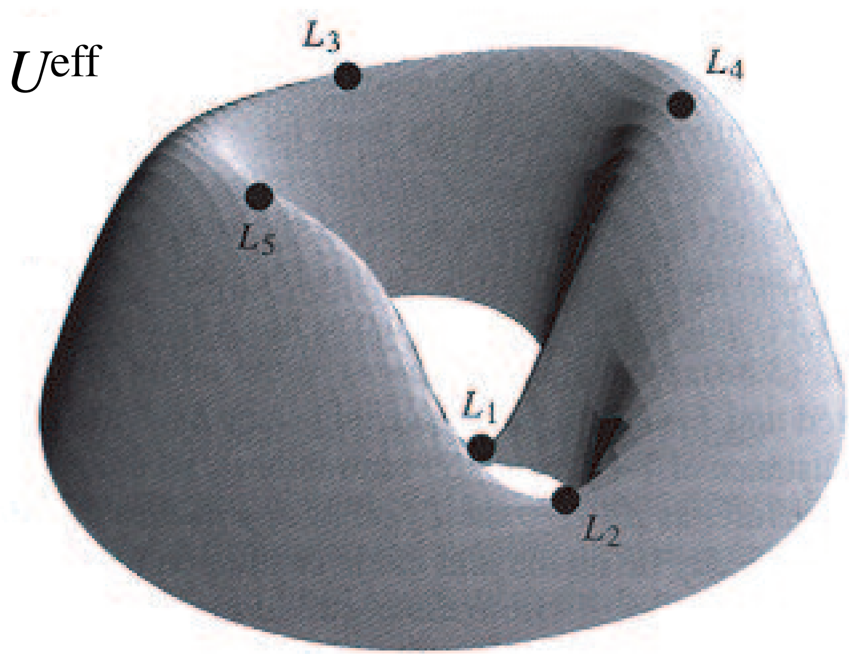
Potential



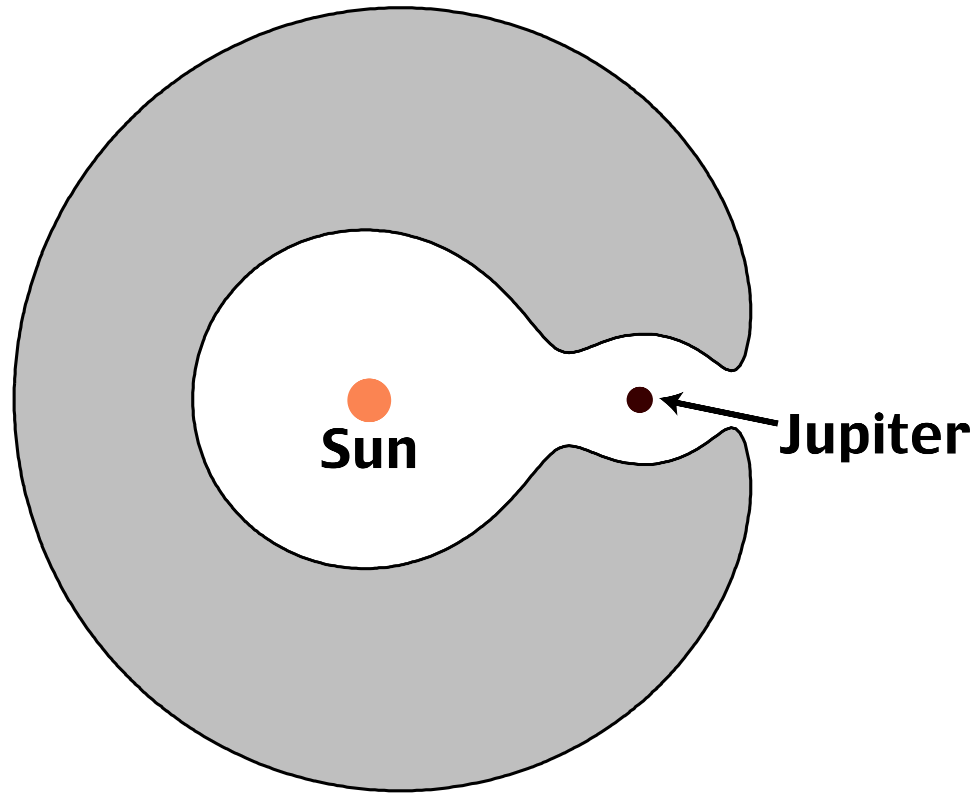
Configuration Space
(Rotating Frame)

Partition the Phase Space

- Comet near the Sun and Jupiter



Potential



Configuration Space
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Partition the Phase Space

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  - We desire a way of describing dynamical boundaries that represent the “frontier” between qualitatively different types of behavior

Partition the Phase Space

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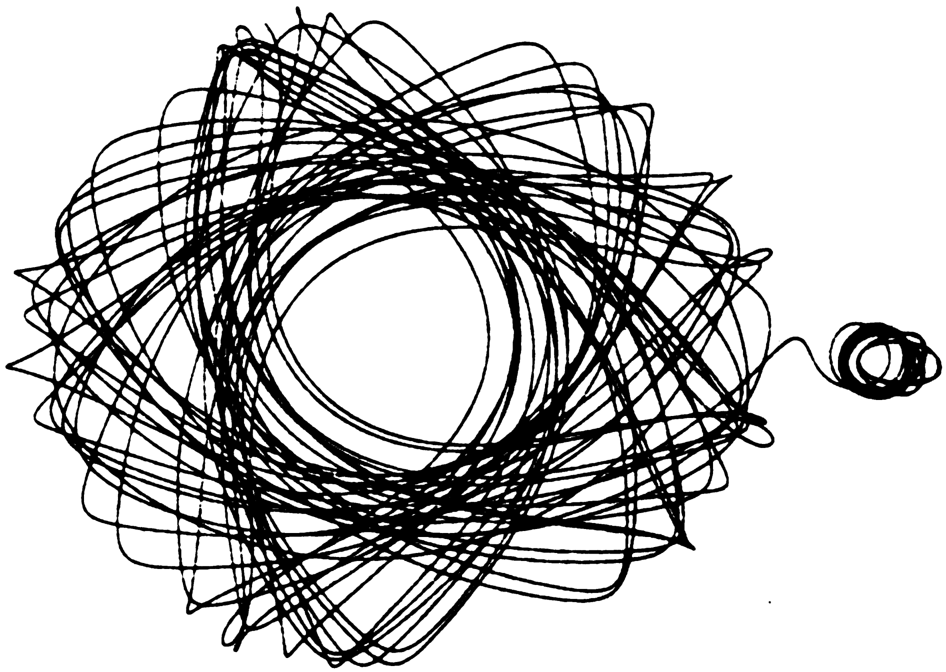
  - We desire a way of describing dynamical boundaries that represent the “frontier” between qualitatively different types of behavior
- ***Example: motion of comet***
  - motion around the Sun
  - motion around Jupiter



Statement of Problem

- Suppose we study the motion on a manifold $\mathcal{M}$
- Suppose $\mathcal{M}$ is partitioned into disjoint regions

$$R_i, i = 1, \dots, N_R,$$

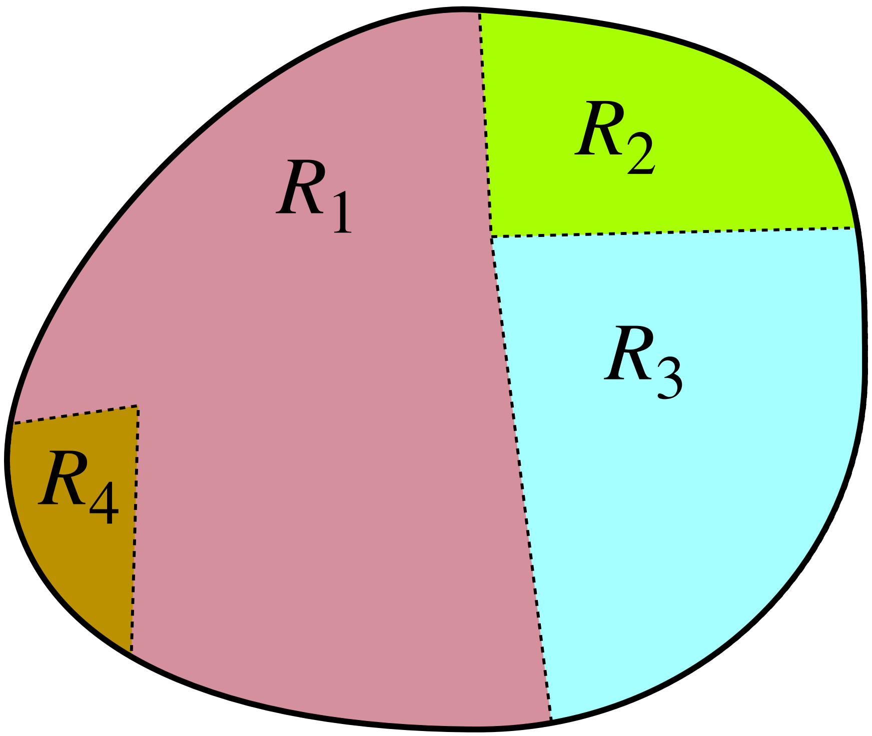
such that

$$\mathcal{M} = \bigcup_{i=1}^{N_R} R_i.$$

- To keep track of the initial condition of a point, we say that *initially* (at $t = 0$) region R_i is uniformly covered with species S_i.
- Thus, species type of a point indicates the region in which it was located initially.

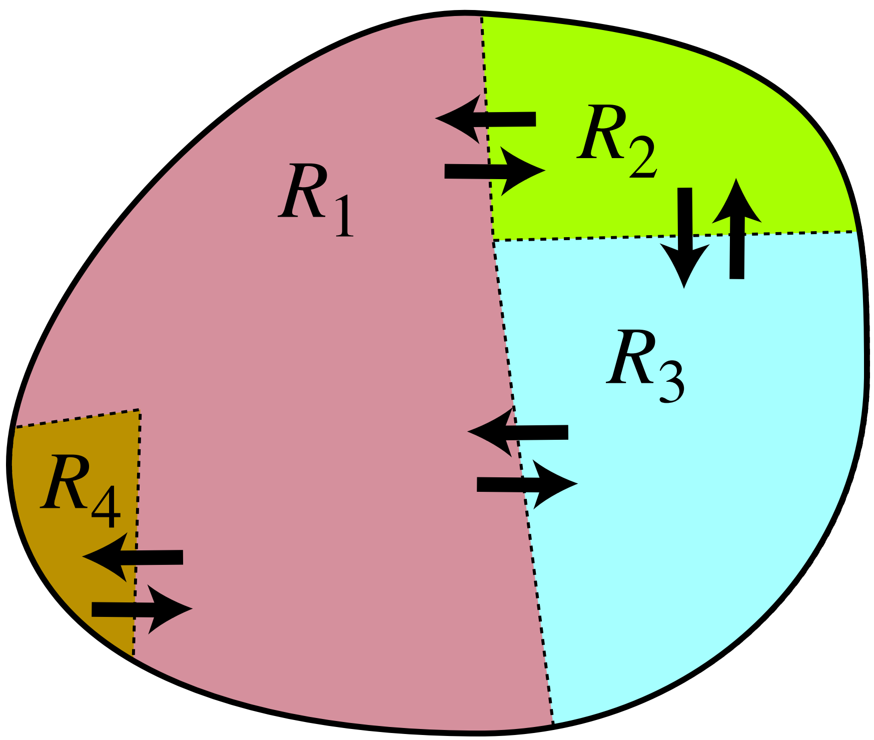
Statement of Problem

- Statement of the transport problem:
  Describe the distribution of species $S_i, i = 1, \ldots, N_R$, throughout the regions $R_j, j = 1, \ldots, N_R$, for any time $t > 0$.



Statement of Problem

- Some quantities we would like to compute are:
  - $T_{i,j}(t)$ = the total amount of species S_i contained in region R_j at time t
  - $F_{i,j}(t) = \frac{dT_{i,j}}{dt}(t)$ = the flux of species S_i into region R_j at time t



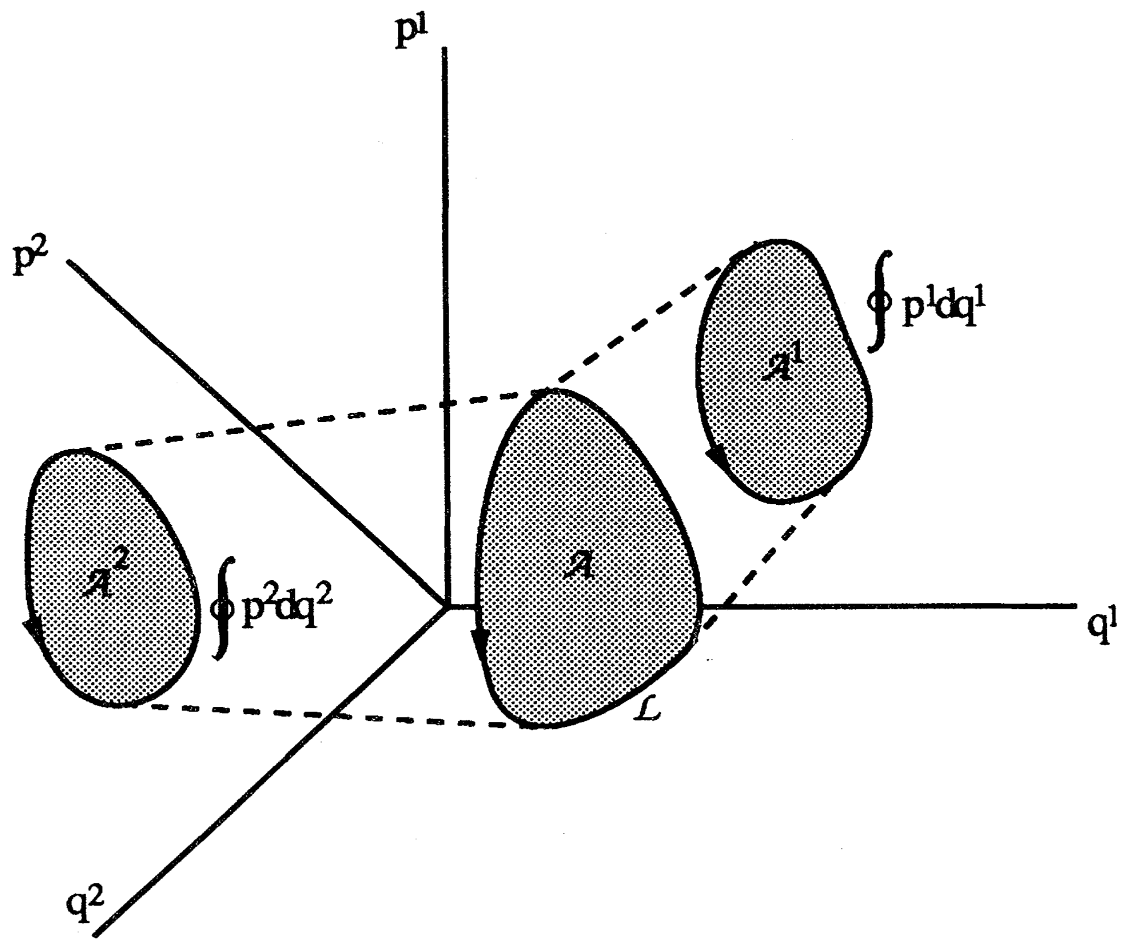
Hamiltonian Systems

- ***Time-independent Hamiltonian*** $H(q,p)$
  - N degrees of freedom
  - Motion constrained to a $(2N-1)$-dimensional energy surface $\mathcal{M}_E$ corresponding to a value $H(q,p) = E = \text{constant}$
  - Symplectic area is conserved along the flow

$$\oint_{\mathcal{L}} p \cdot dq = \int_{\mathcal{A}} dp \wedge dq = \text{constant}$$

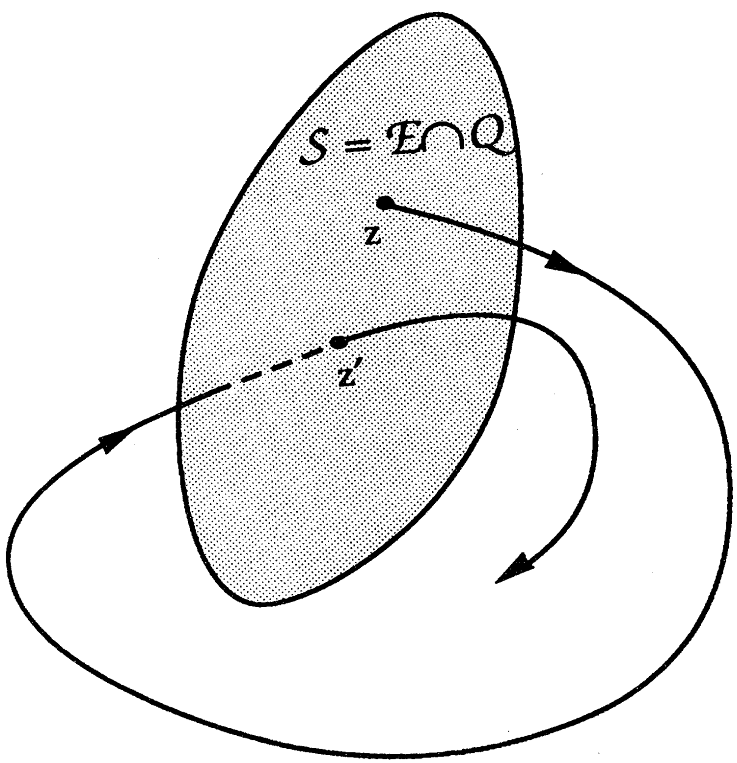
Symplectic Area Conserved

$$\sum_{i=1}^{N} \sigma_i \int_{\mathcal{A}^i} dp_i dq^i = \text{constant on an energy surface}$$



Poincaré Section

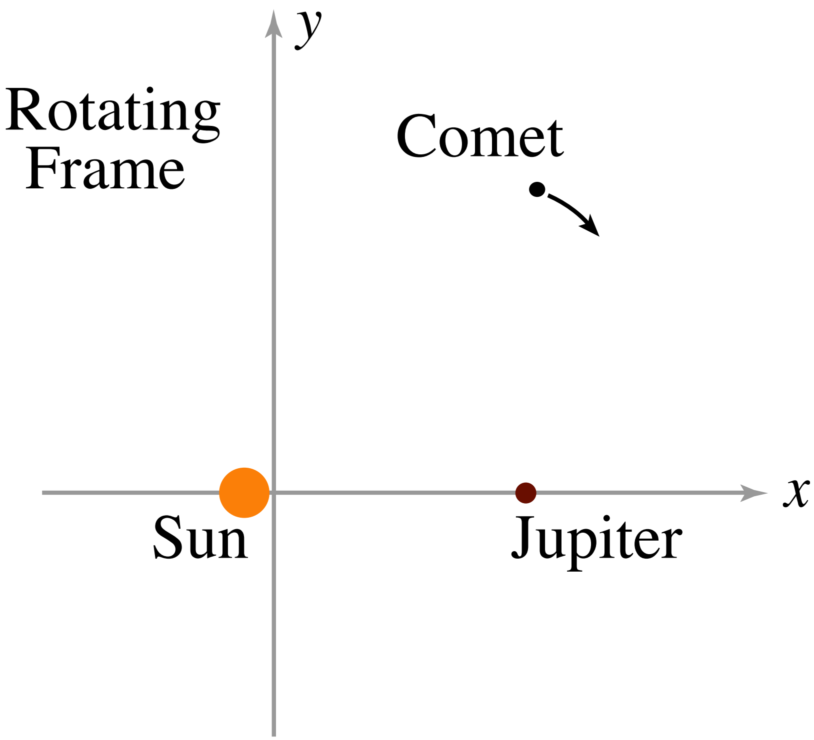
- Suppose there is another $(2N-1)$-dimensional surface $\mathcal{Q}$ that is transverse (i.e., nowhere parallel) to the flow in some local region.
- The Poincaré section $\mathcal{S}$ is the $(2N-2)$-dimensional intersection of $\mathcal{M}_E$ with $\mathcal{Q}$.



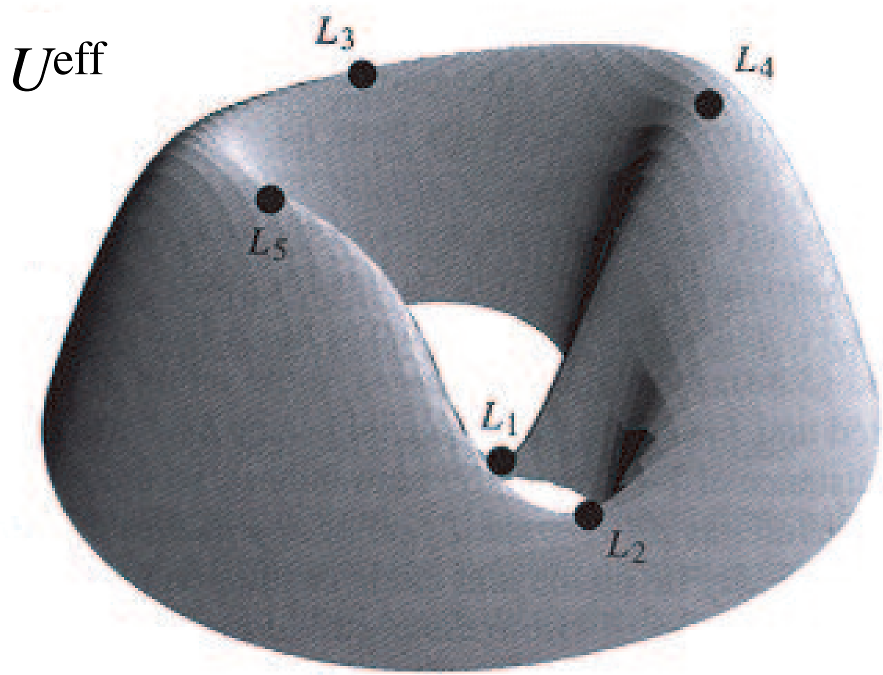
Example for $N = 2$

- ***Circular restricted 3-body prob. (2D)***

$$H = \frac{1}{2}((p_x + y)^2 + (p_y - x)^2) + U^{\mathrm{eff}}(x, y)$$



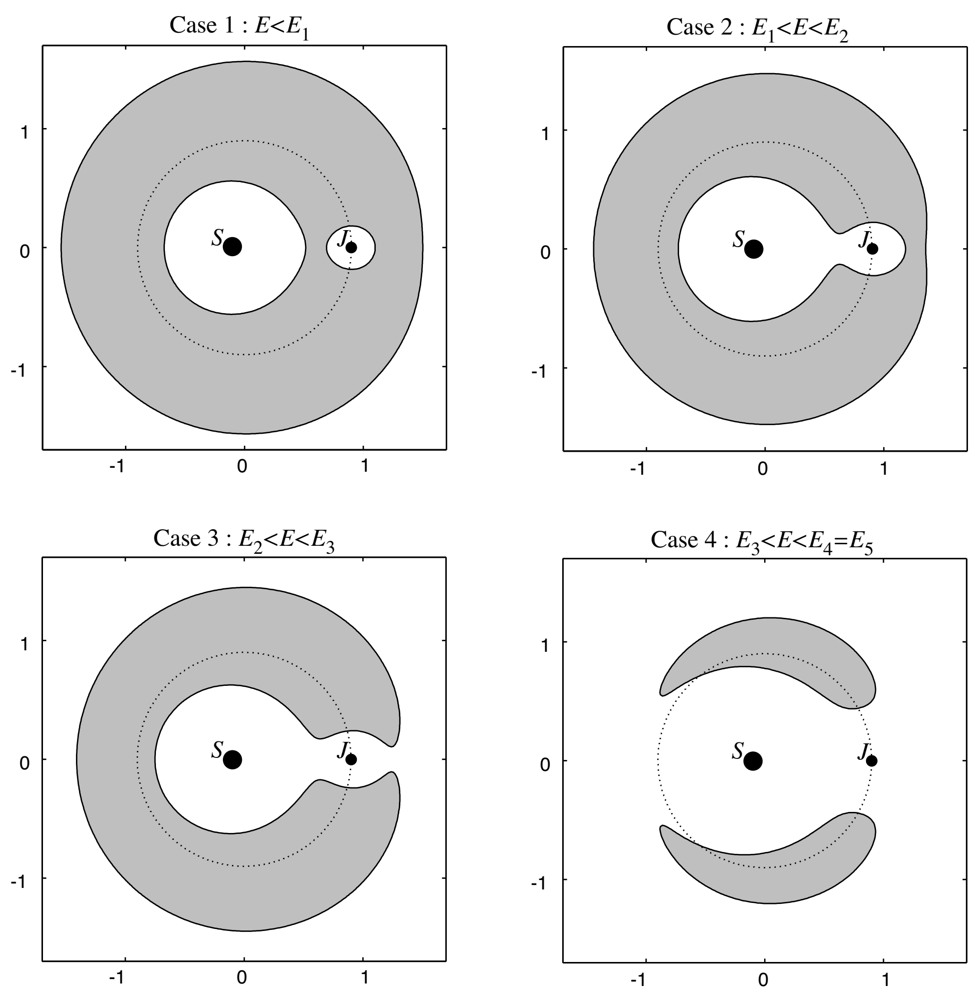
Position Space



Effective Potential

3-Body Problem (2D)

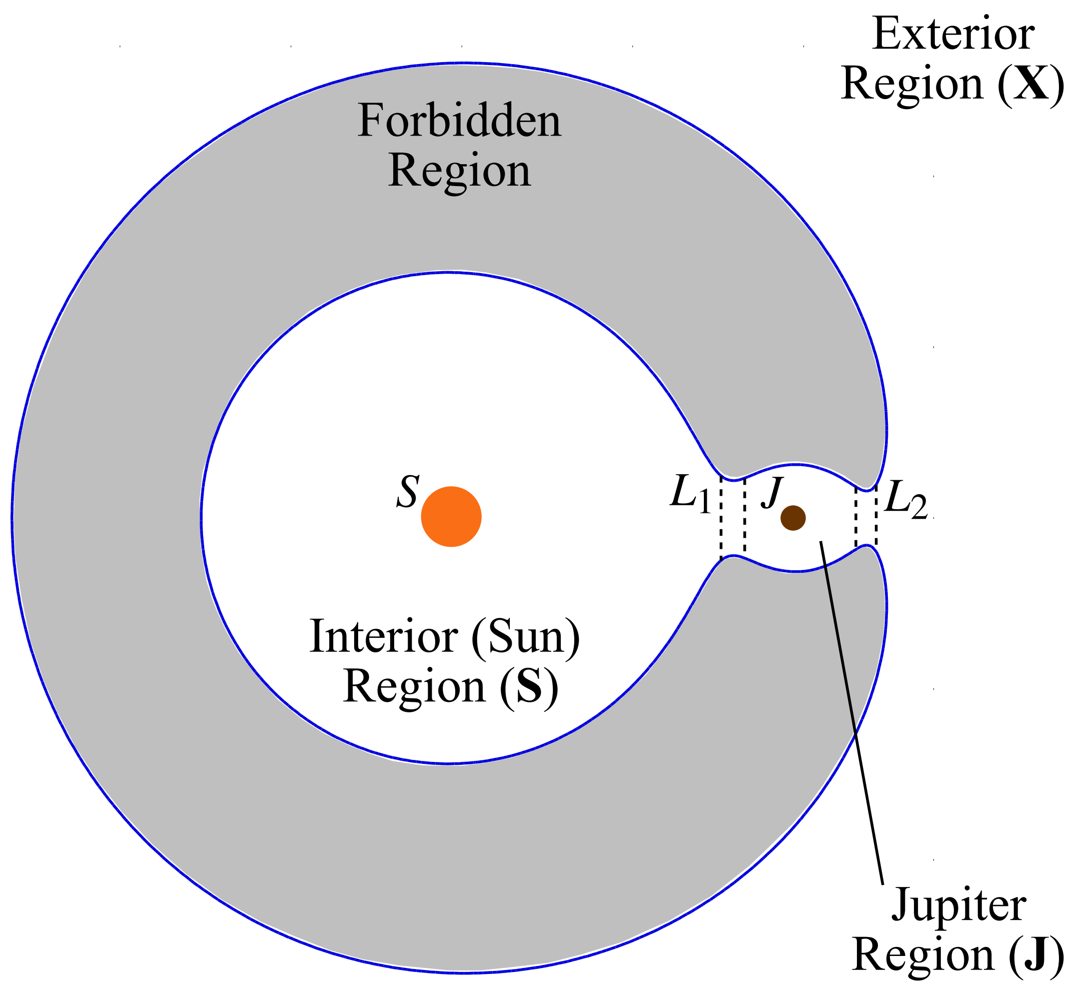
■ *Look at fixed energy*



Position Space Projections

3-Body Problem (2D)

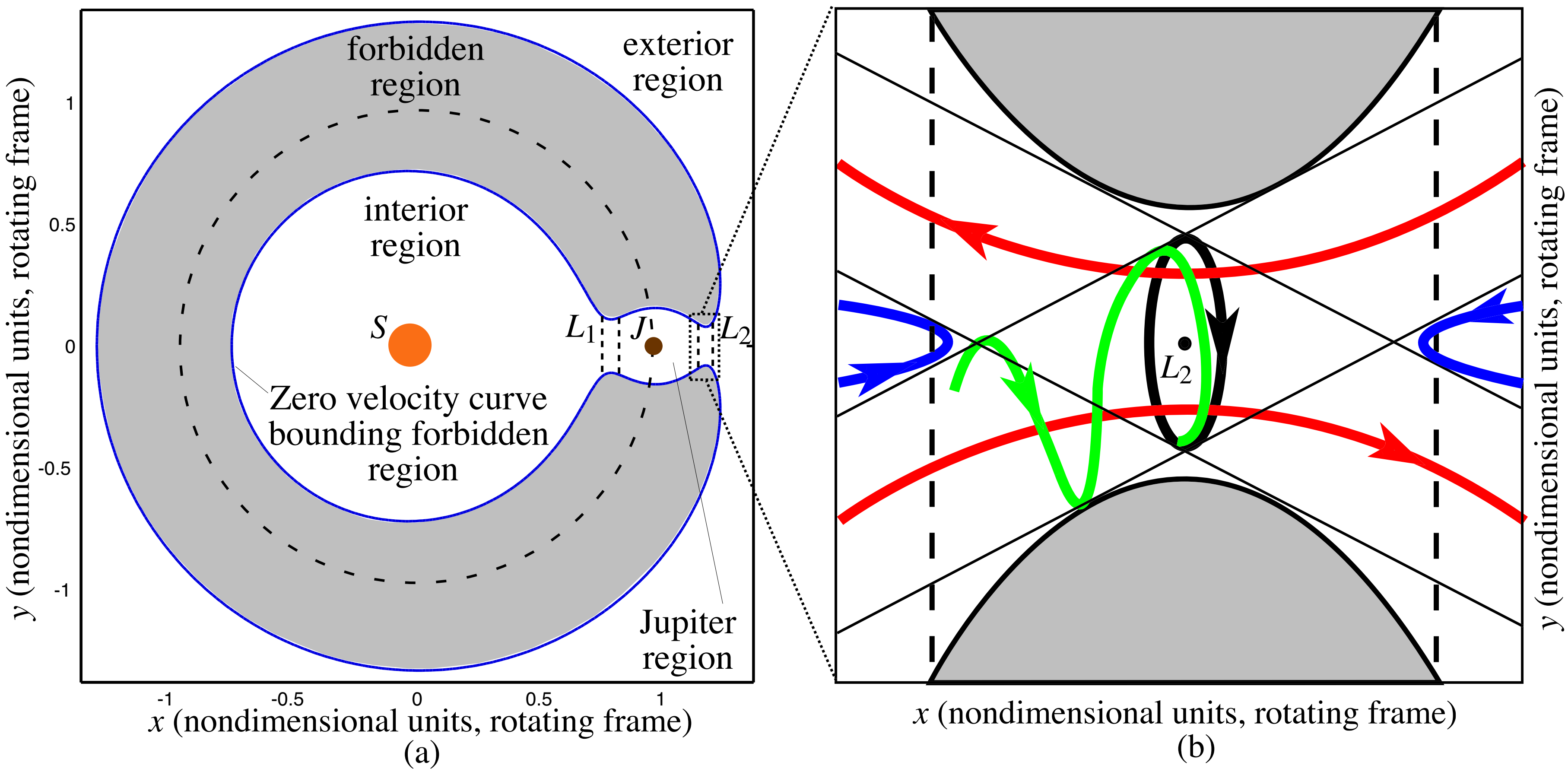
- ***Partition the energy surface***



Position Space Projection

3-Body Problem (2D)

- ***Look at motion near "saddle points"***



Position Space Projection

Potential Barriers

- Hamiltonian systems with potential barriers give rise to "saddle points" whose local form is given by

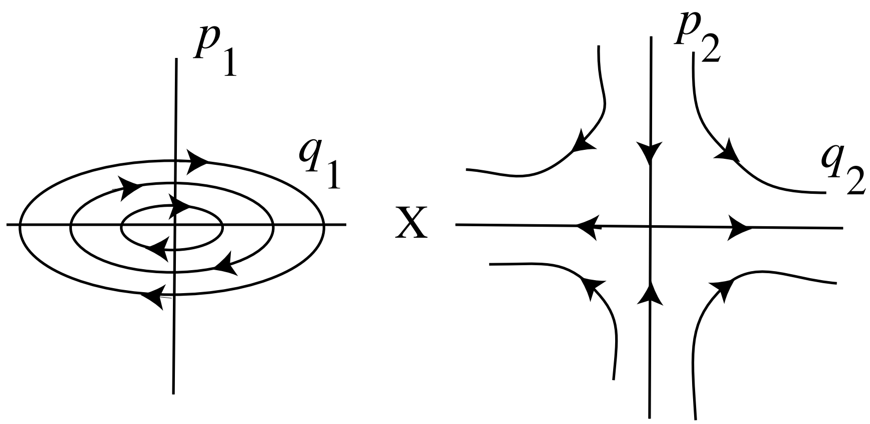
$$H(q,p) = \frac{\omega}{2}(q_1^2 + p_1^2) + \lambda q_2 p_2, \tag{1}$$

i.e., linearized vector field has eigenvalues $\pm i\omega$, $\pm\lambda$.

- Moser [1958] showed that the qualitative behavior of (1) carries over to the full nonlinear equations.
- In particular, the flow of (1) has form center $\times$ saddle.

Local Dynamics

- For fixed energy $H = h$, energy surface $\simeq S^2 \times \mathbb{R}$.
- Other constants of motion: $I_1 = q_1^2 + p_1^2$ and $I_2 = q_2 p_2$.



- Normally hyperbolic invariant manifold at $q_2 = p_2 = 0$, i.e.,

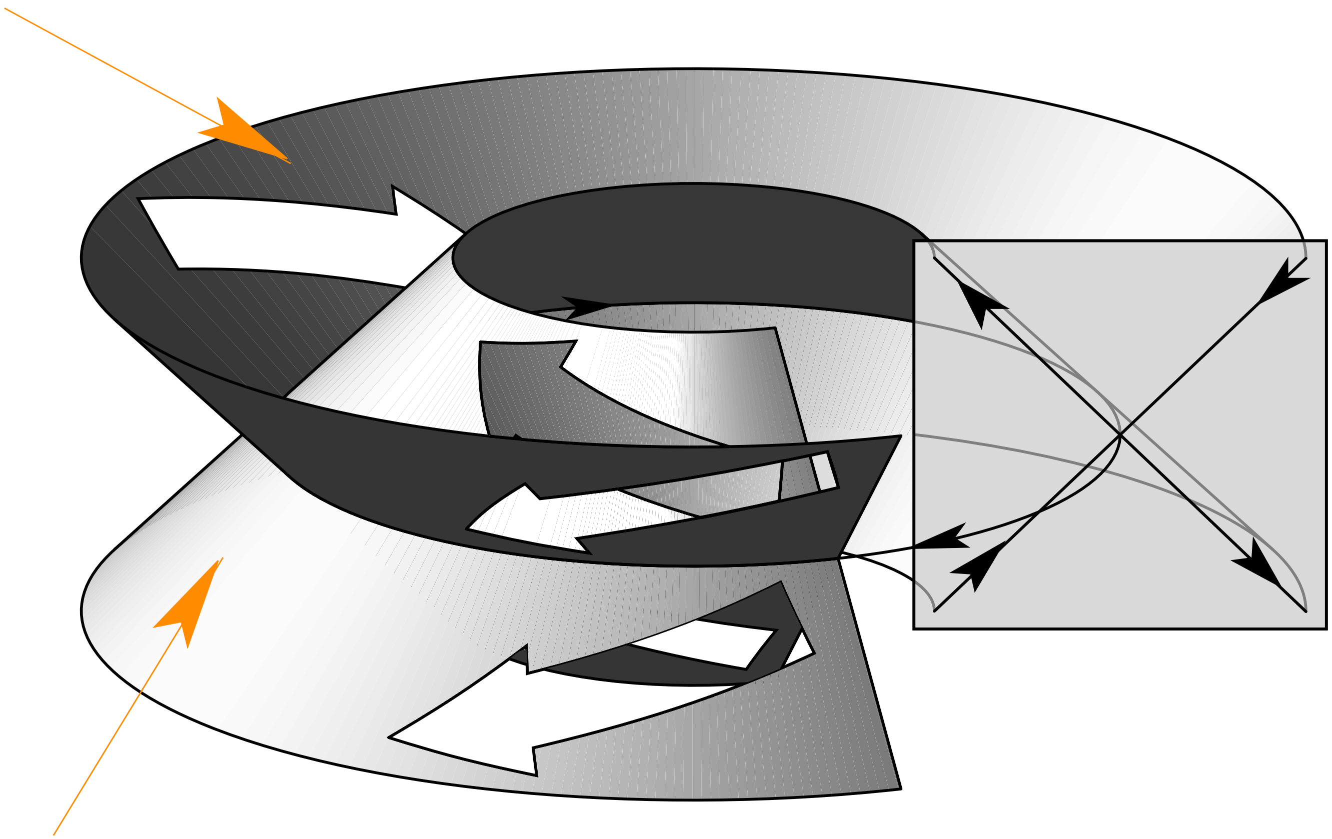
$$\mathcal{M}_h = \frac{\omega}{2}(q_1^2 + p_1^2) = h > 0.$$

Note that $\mathcal{M}_h \simeq S^1$, a periodic orbit.

Local Dynamics

- Four cylinders of asymptotic orbits: the stable and unstable manifolds $W^s_\pm(\mathcal{M}_h), W^u_\pm(\mathcal{M}_h)$.

Stable Manifold (orbits move toward the periodic orbit)



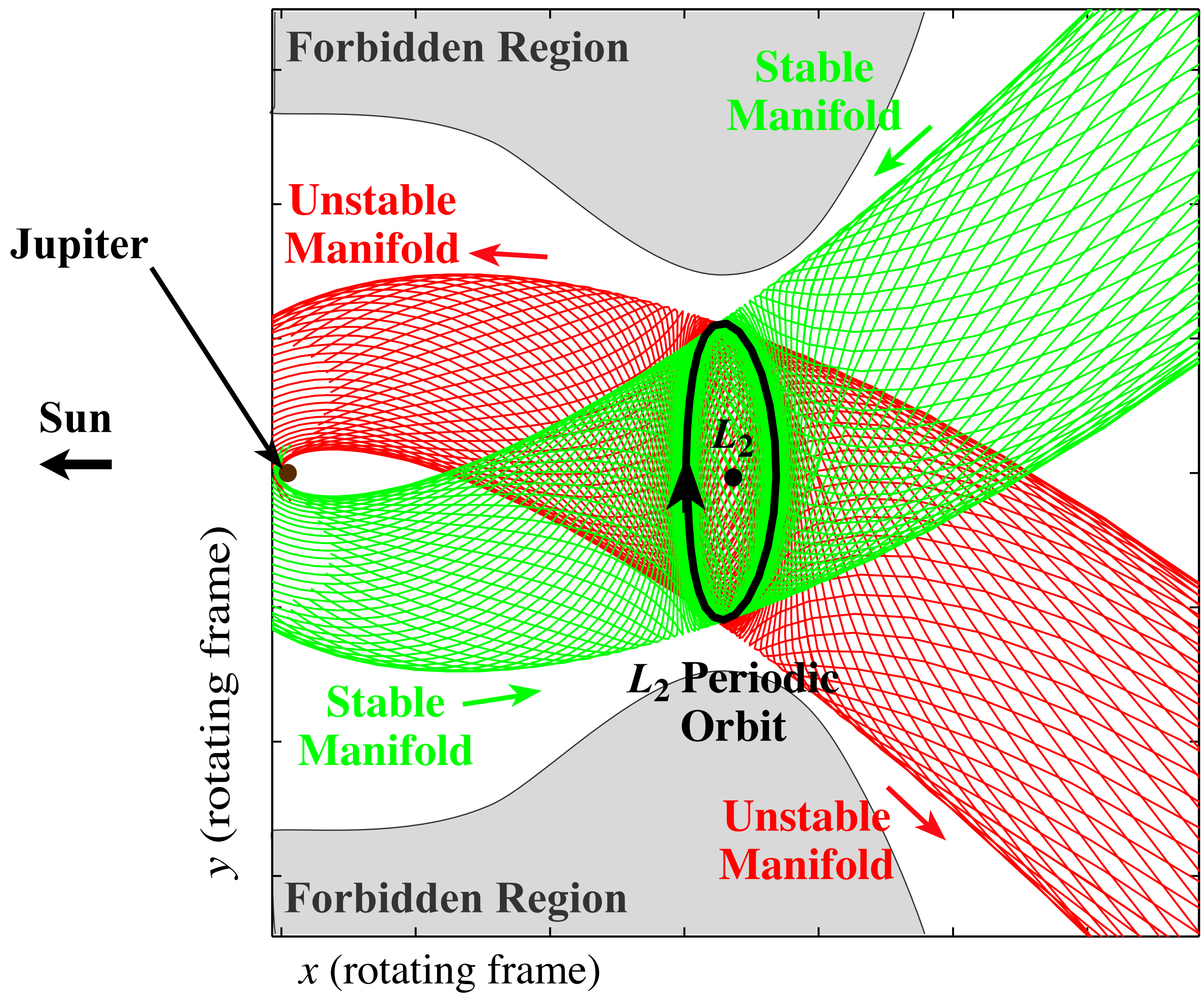
Unstable Manifold (orbits move away from the periodic orbit)

Transit and Nontransit Orbits

- Cylinders separate transit from nontransit orbits.
- In three-body problem:
  - These manifold tubes play an important role in what passes by Jupiter (transit orbits)
  - and what bounces back (non-transit orbits)
  - transit possible for objects "inside" the tube, otherwise no transit — this is important for transport issues

Tubes in the 3-Body Problem

- Stable and unstable manifold tubes
  - Control transport through the potential barrier.



Flux

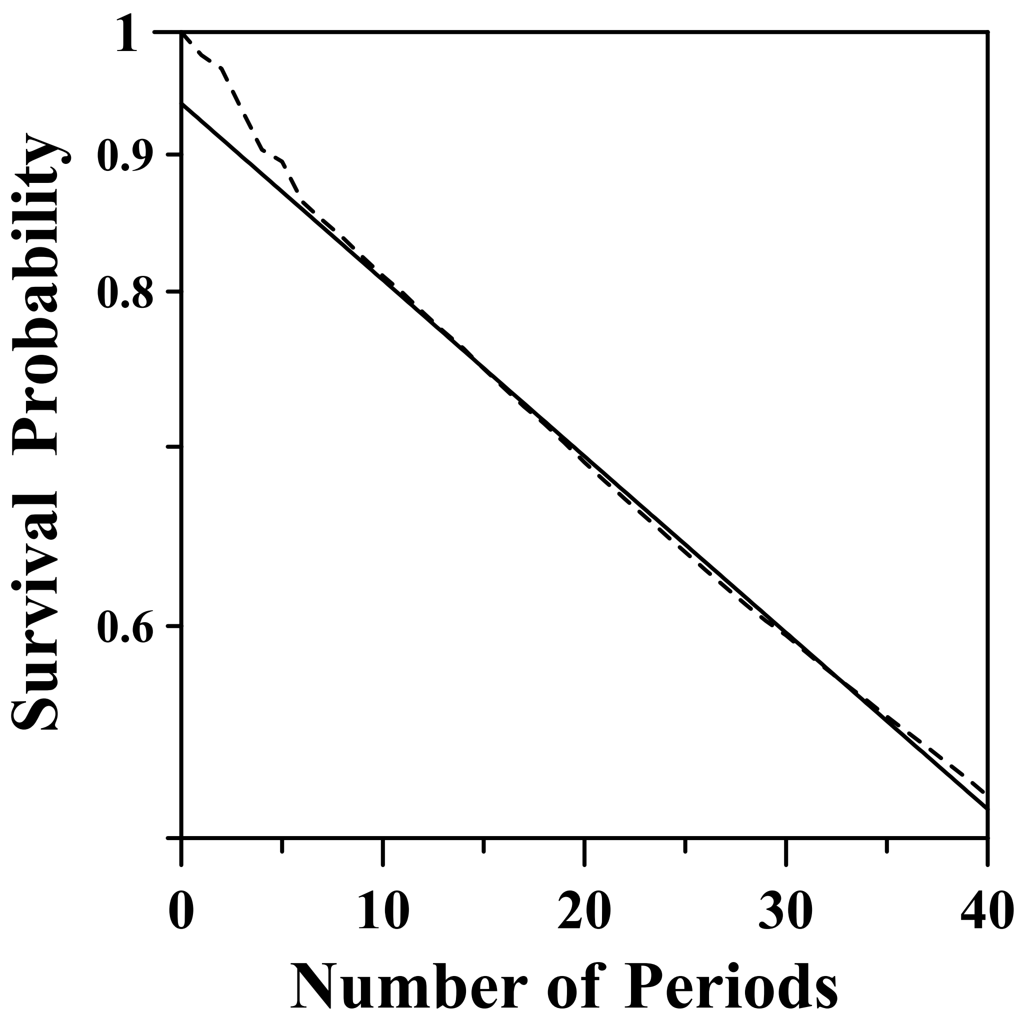
- ***Tubes of transit orbits are the relevant objects to study***
  - Tubes determine the flux between regions $F_{i,j}(t)$.
  - Note, net flux is zero for volume-preserving motion, so we consider the one-way flux.
  - Example: $F_{J,S}(t) =$ volume of trajectories that escape from the Jupiter region into the Sun region per unit time.

Transition Probabilities

- ***Fluxes give rates and probabilities***
  - Recently, Jaffé, Ross, Lo, Marsden, Farrelly, and Uzer [2002] computed the rate of escape of asteroids temporarily captured by Mars.
  - Theory and numerical simulations agree well.

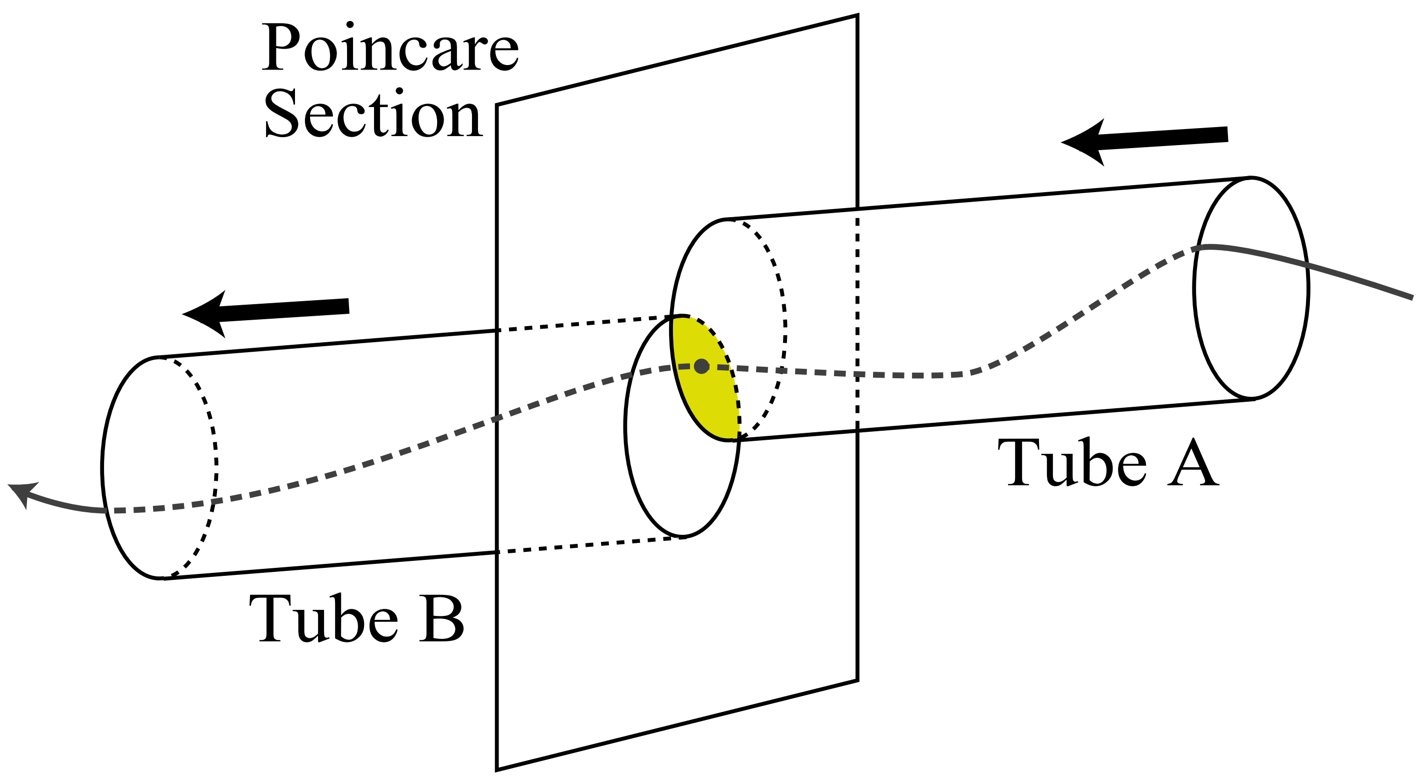
Transition Probabilities

- Monte Carlo simultion (dashed) and theory (solid)



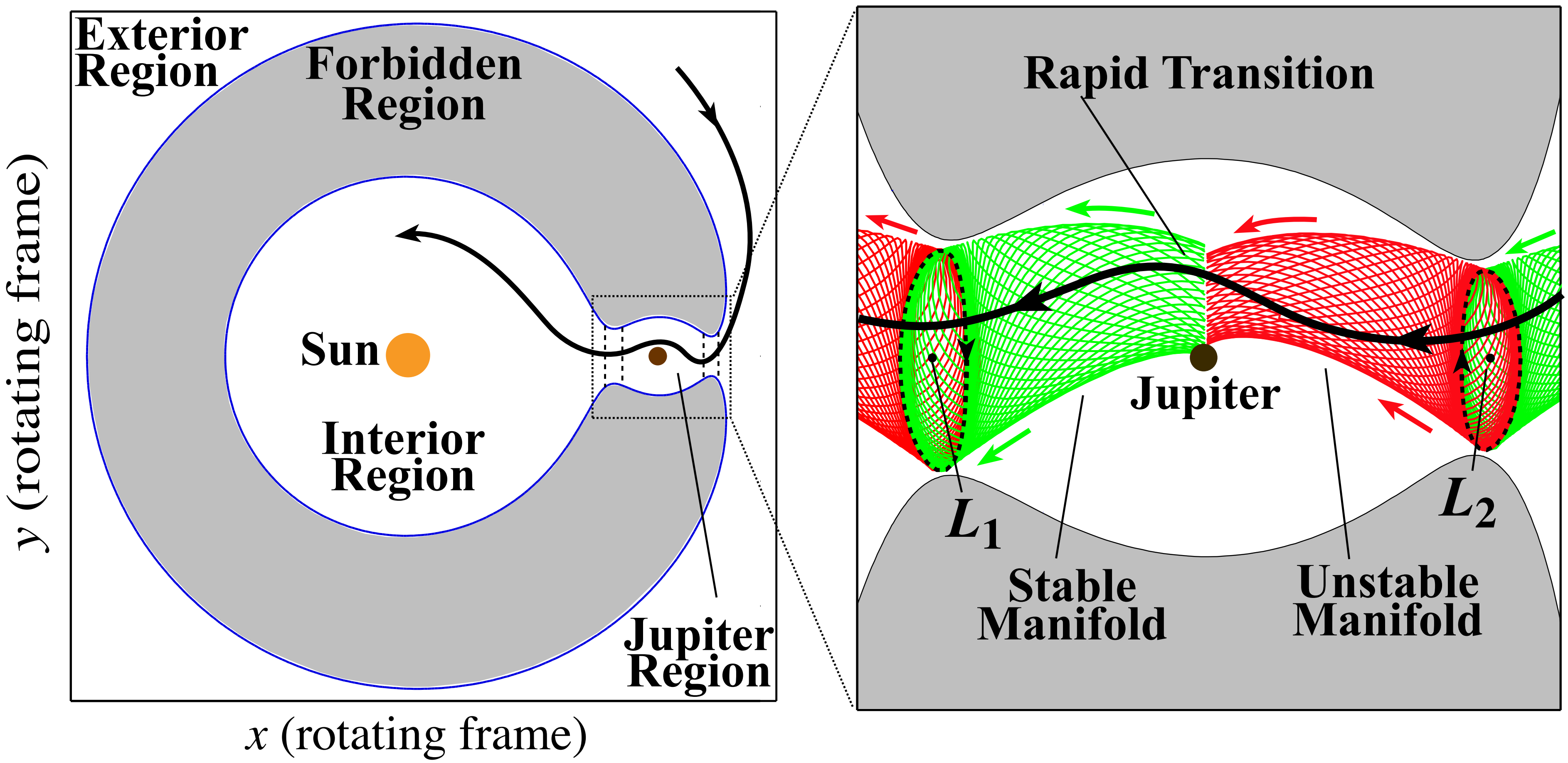
Transition Probabilities

- *More exotic transport between regions*
  - Look at the intersections between the interior of stable and unstable tubes on the same energy surface.
  - Could be from different potential barrier saddles.



Transition Probabilities

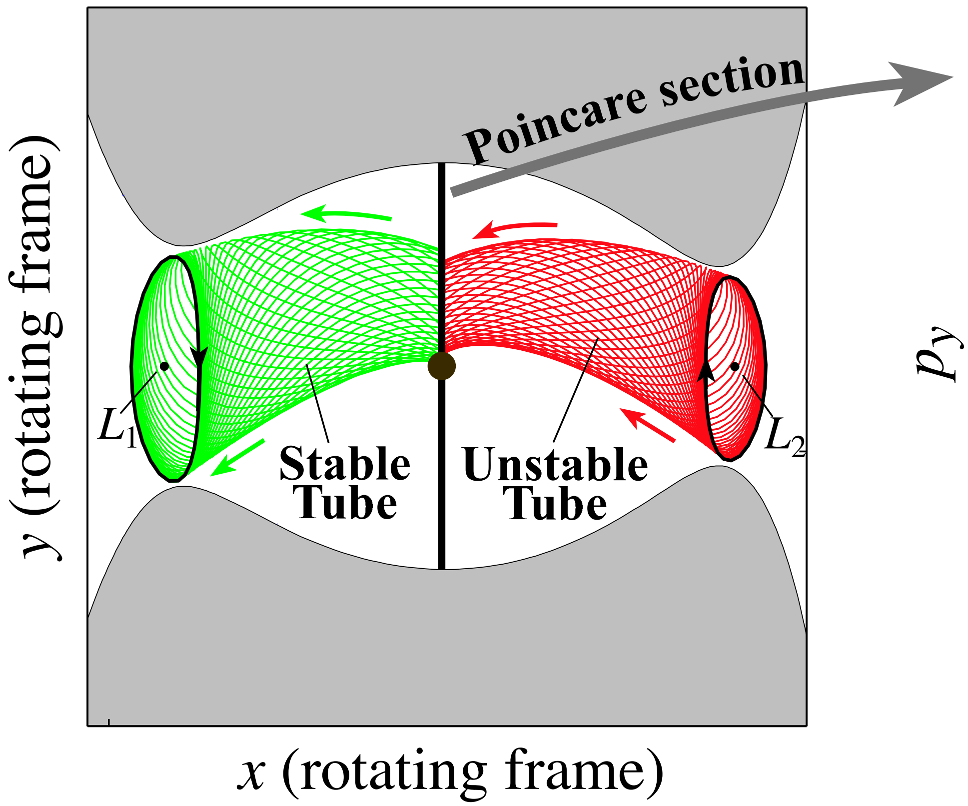
- Example: Comet transport between outside and inside of Jupiter



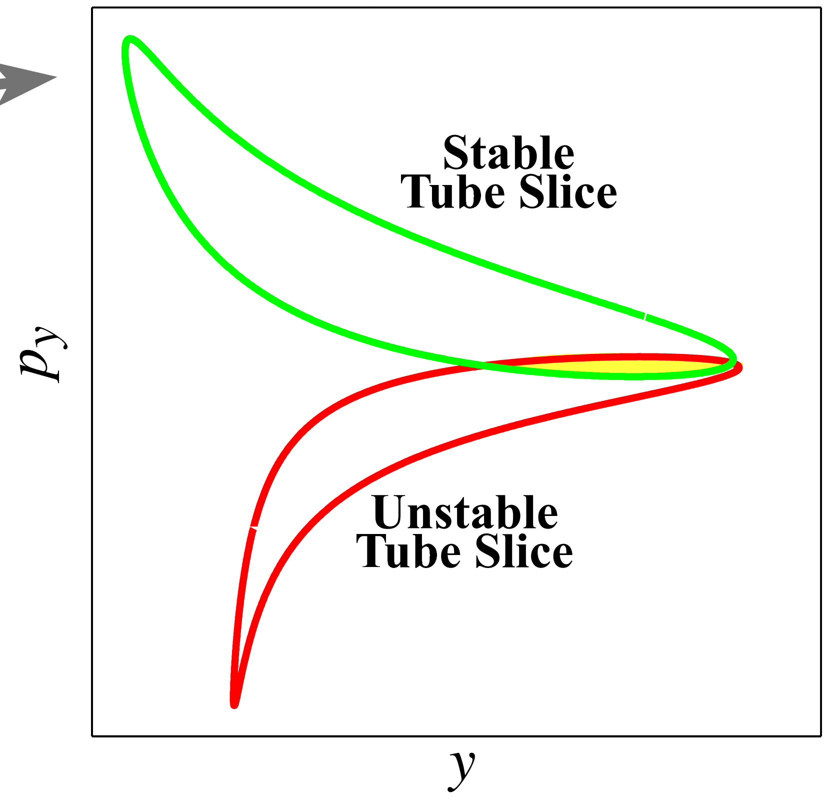
(a)

Transition Probabilities

- Look at Poincaré section intersected by both tubes.
- Choosing surface $\{x = \text{constant}; p_x < 0\}$, we look at the canonical plane (y, p_y).



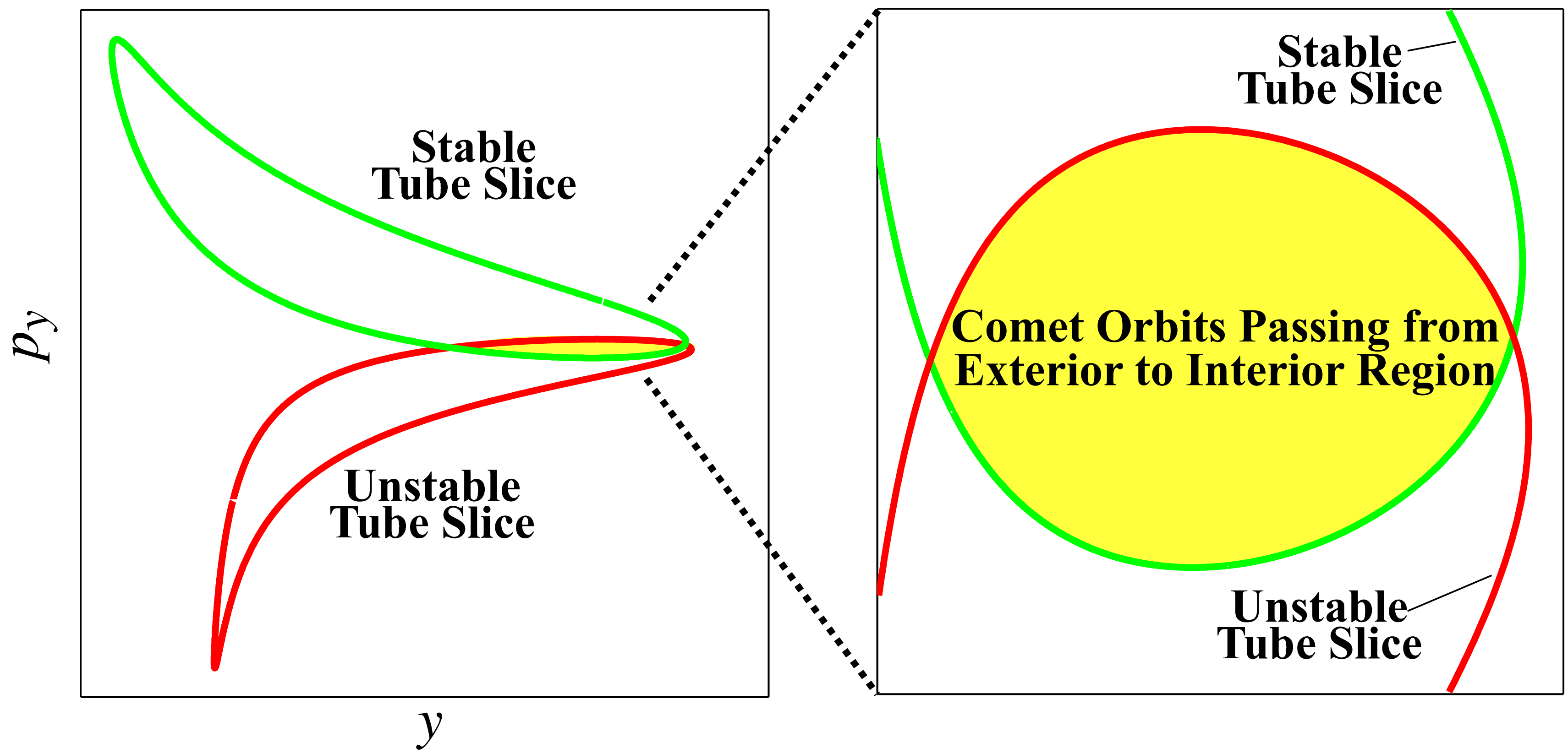
Position Space



Canonical Plane (y, p_y)

Transition Probabilities

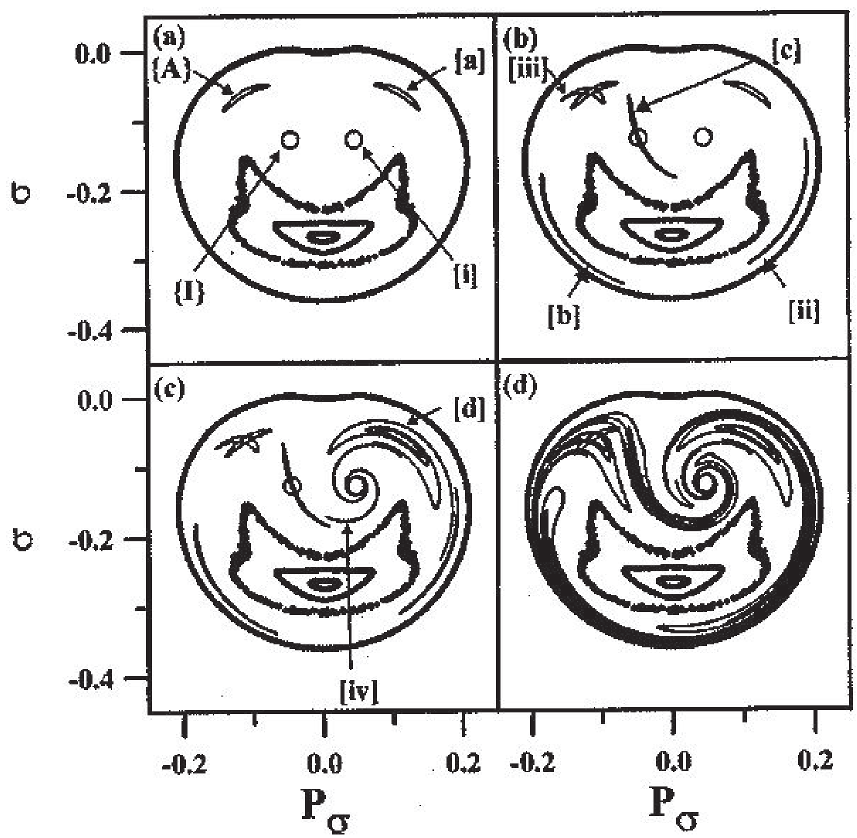
- Relative canonical area gives relative volume of orbits.
- Can be interpreted as the probability of transition from one region to another.



Canonical Plane (y, p_y)

Mixing

- By keeping track of the intersections of the tubes, one can describe the mixing of different regions ($T_{i,j}(t)$).
  - It can get messy fast!



(from Jaffé, Farrelly and Uzer [1999])

Some Notes on Computation

- ***Computationally challenging!***
- ***Periodic orbits***
  - high order analytic expansion (see Llibre et al., 1985)
  - normal form theory
  - numerical continuation (AUTO2000 software)

Some Notes on Computation

Stable and unstable manifolds

- Suppose ODE in $\mathbb{R}^n$ of form

$$\dot{x} = f(x)$$

  with periodic solution $\bar{x}(t)$ of period T.

- The **variational equations** are linearized equations for variations $\delta\bar{x}$ about $\bar{x}$:

$$\begin{aligned} \dot{\delta\bar{x}}(t) &= Df(\bar{x}(t))\delta\bar{x}(t) \\ &= A(t)\delta\bar{x}(t), \end{aligned}$$

  where $A(t)$ is an $n \times n$ matrix of period T.

Some Notes on Computation

- Solutions are known to be of the form

$$\delta\bar{x}(t) = \Phi(t,0)\delta\bar{x}(0),$$

where $\Phi(t,0)$ is the **state transition matrix** (STM) from time 0 to t.

- The STM along a reference orbit is computed by numerically integrating $n(n+1)$ ODEs:

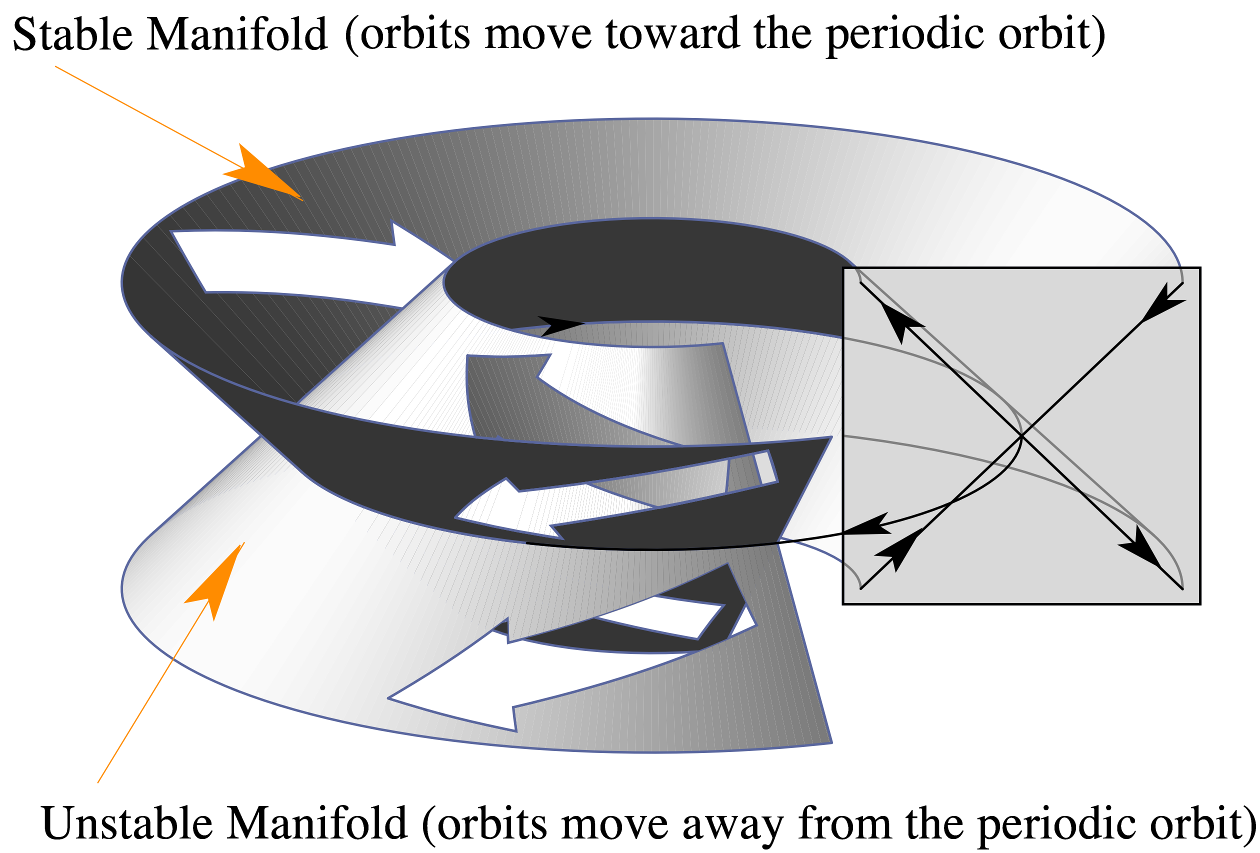
$$\dot{\bar{x}} = f(\bar{x}),$$
$$\dot{\Phi}(t,0) = A(t)\Phi(t,0),$$

with initial conditions:

$$\bar{x}(0) = \bar{x}_0,$$
$$\Phi(0,0) = I_6.$$

Some Notes on Computation

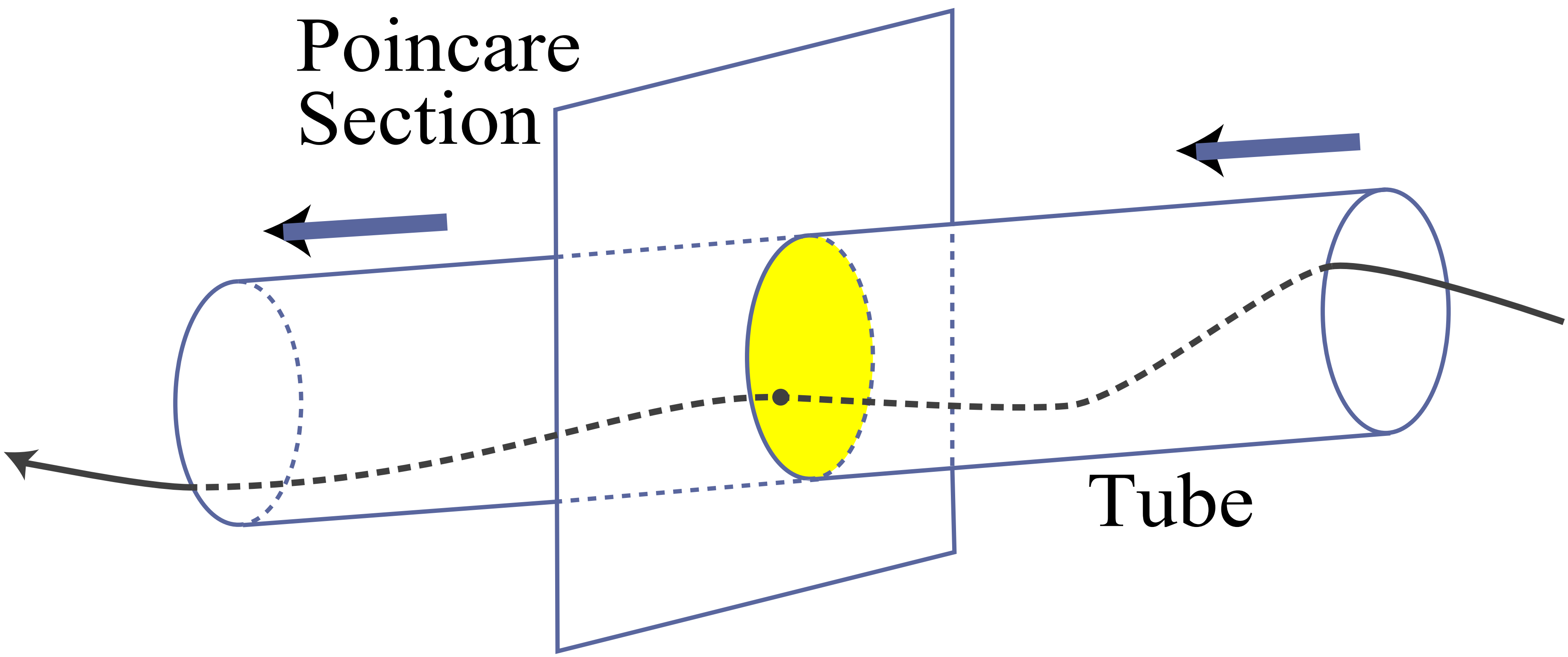
- The **monodromy matrix** $\Phi(T,0)$ has an unstable and stable eigenvector. We can numerically integrate this linear approximation to the unstable (or stable) direction to obtain the unstable (or stable) manifold.



Some Notes on Computation

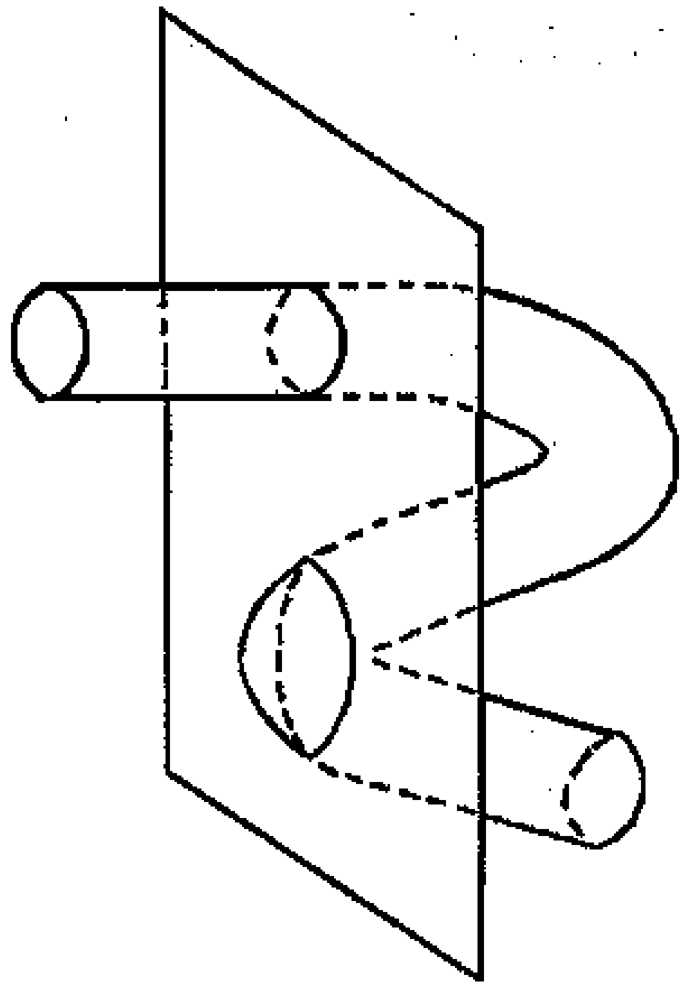
Poincaré Sections

- This set of solutions approximating the unstable manifold can be numerically integrated until some stopping condition is reached (e.g., $x_j = \mathrm{constant}$).



Some Notes on Computation

- Problems:
  How to handle non-transversal intersections



$N = 3$ or More

- ***Extend to $N \geq 3$ degrees of freedom***
  - Near equilibrium point, suppose linearized Hamiltonian vector field has eigenvalues $\pm i\omega_j$, $j = 1, \ldots, N-1$, and $\pm\lambda$.
  - Assume the complexification is diagonalizable.
  - Hamiltonian normal form theory tranforms Hamiltonian into a lowest order form:

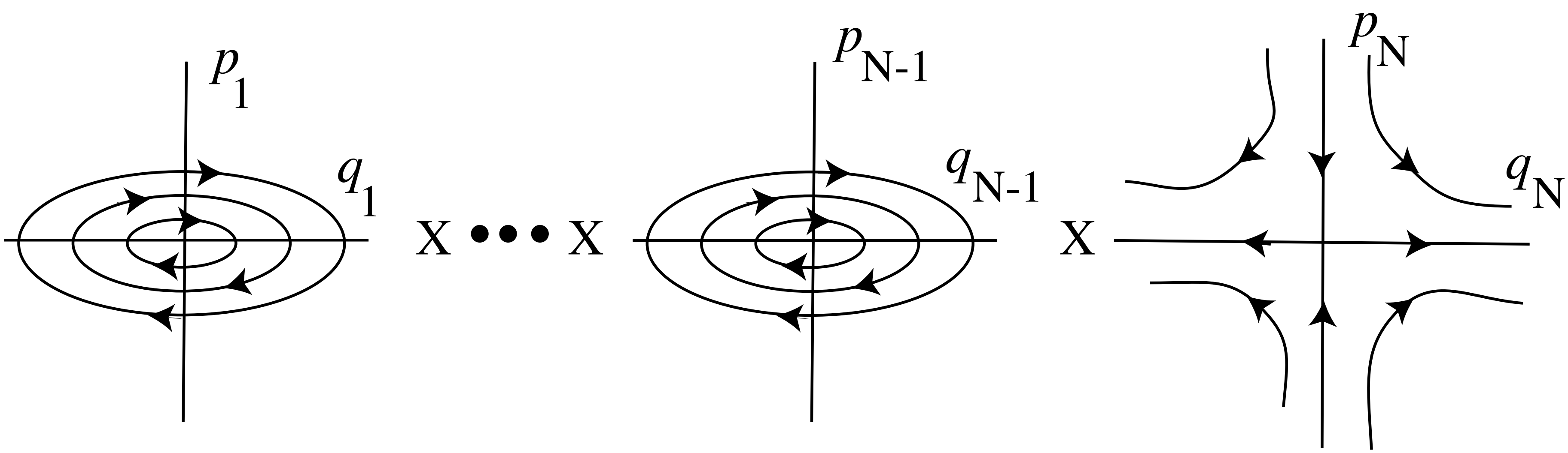
$$H(q,p) = \sum_{i=1}^{N-1} \frac{\omega_i}{2}\left(p_i^2 + q_i^2\right) + \lambda q_N p_N.$$

  - Equilibrium point is of type center $\times \cdots \times$ center $\times$ saddle ($N-1$ centers).

$N = 3$ or More

Multidimensional "saddle point"

- For fixed energy $H = h$, energy surface $\simeq S^{2N-2} \times \mathbb{R}$.
- Constants of motion:
  $I_j = q_j^2 + p_j^2, j = 1, \ldots, N-1$, and $I_N = q_N p_N$.



The $\boldsymbol{N}$ Canonical Planes

$N = 3$ or More

- Normally hyperbolic invariant manifold at $q_N = p_N = 0$,

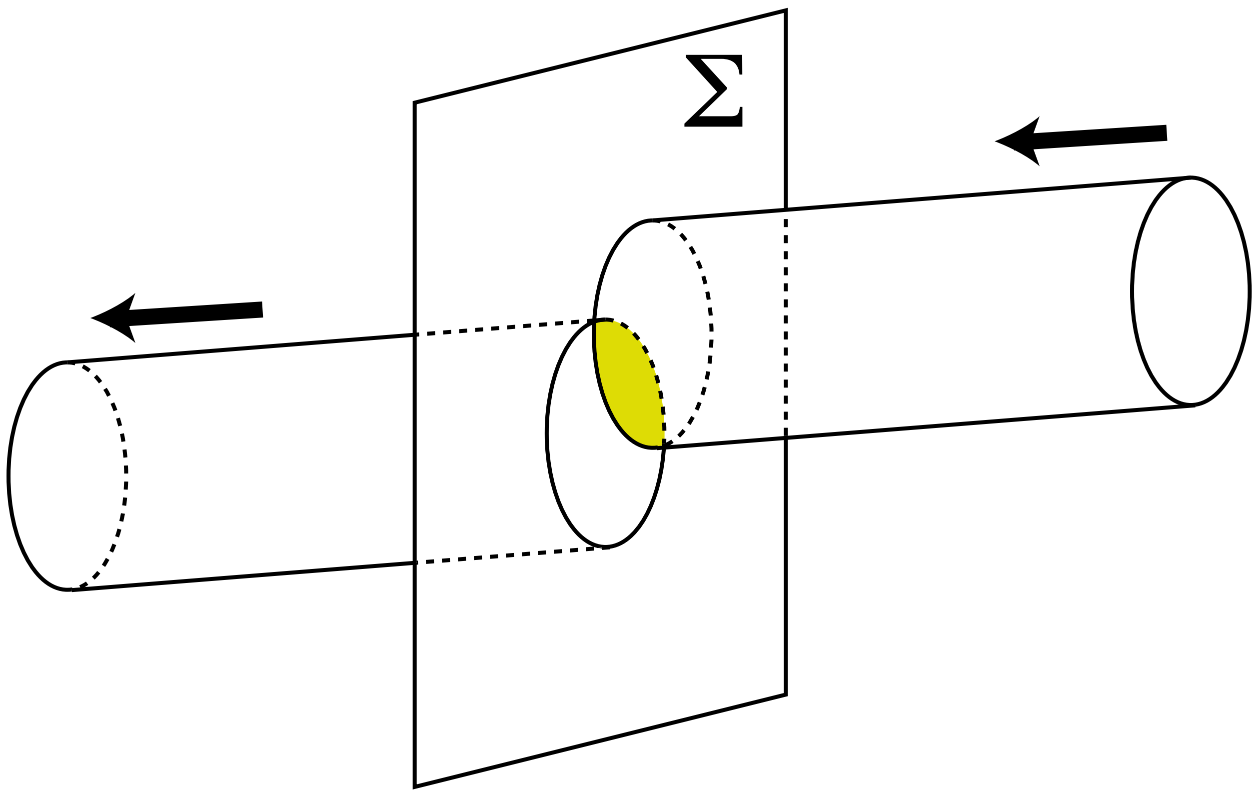
$$\mathcal{M}_h = \sum_{i=1}^{n-1} \frac{\omega_i}{2} \left(p_i^2 + q_i^2\right) = h > 0.$$

  Note that $\mathcal{M}_h \simeq S^{2N-3}$, not a single trajectory.

- Four "cylinders" of asymptotic orbits: the stable and unstable manifolds $W^s_\pm(\mathcal{M}_h), W^u_\pm(\mathcal{M}_h)$, which have the structure $S^{2N-3} \times \mathbb{R}$.

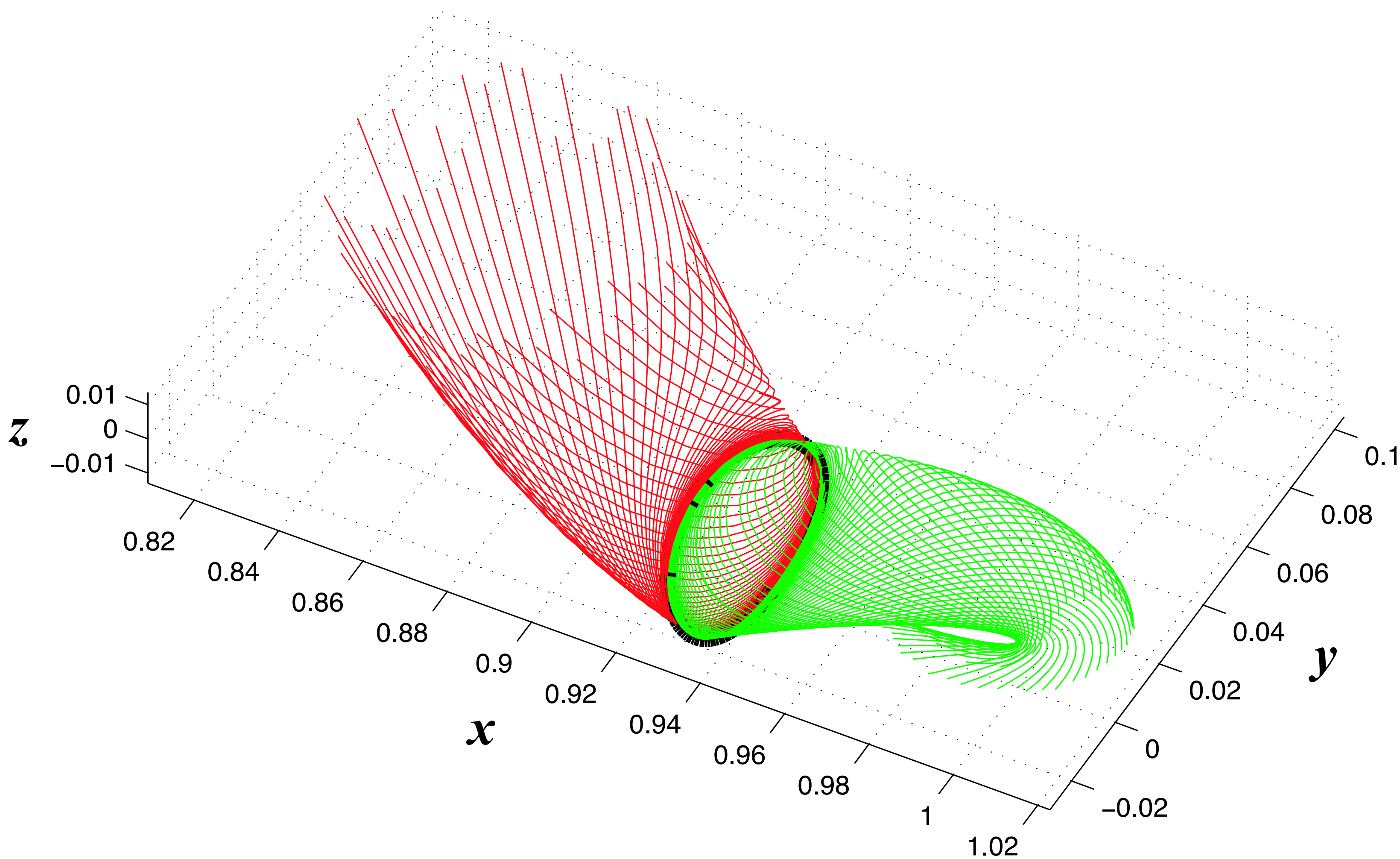
$N = 3$ or More

- Transport between regions is mediated by the "higher dimensional tubes"
- Compute fluxes, transition probabilities, etc.



$N = 3$ or More

- Example: restricted three-body problem (3D)



3D Position Space

Future Directions

Future Directions

- Compute fluxes, transition probabilities in 2 and 3 degree of freedom systems
- Add small dissipation
  - Can Hamiltonian methods still be used?
- Determine statistical laws for astronomical systems
  - Over a range of energies
  - Is ergodic assumption valid?
  - Obtain useful asteroid collision probabilities, etc.
- Combine with control for spacecraft navigation

References

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