Experimental validation of phase space conduits of transition between potential wells

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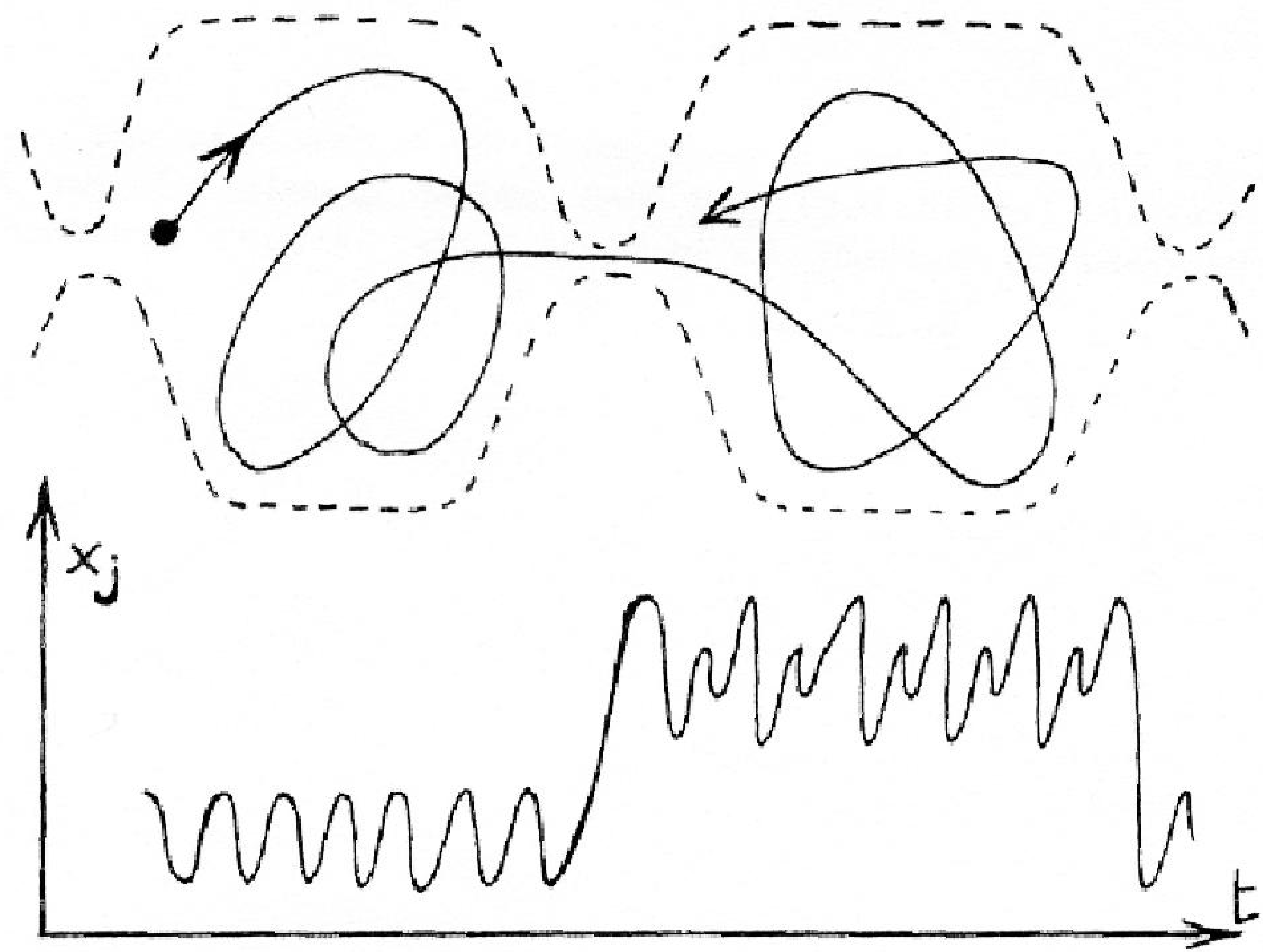
Intermittency and chaotic transitions

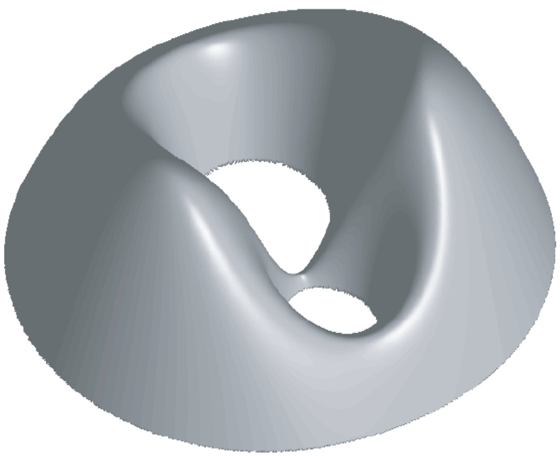
e.g., escaping or transitioning through "bottlenecks" in phase space

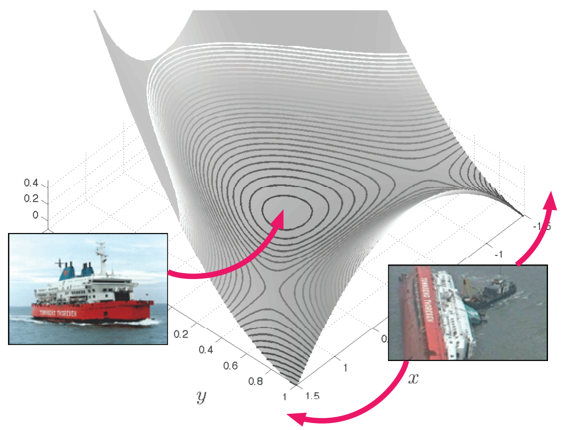


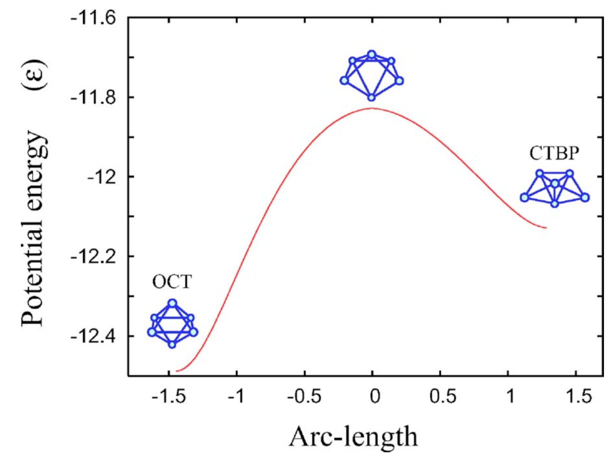
Marchal [1990]

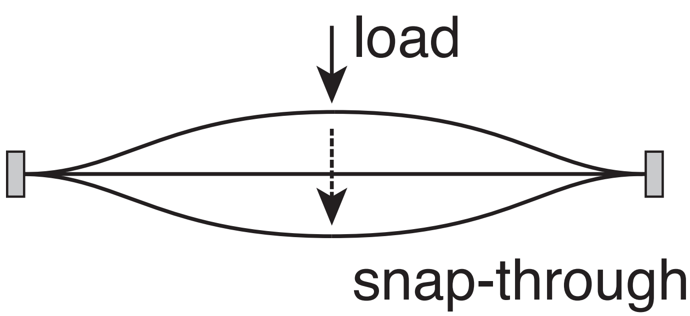
Multi-well multi-degree of freedom systems

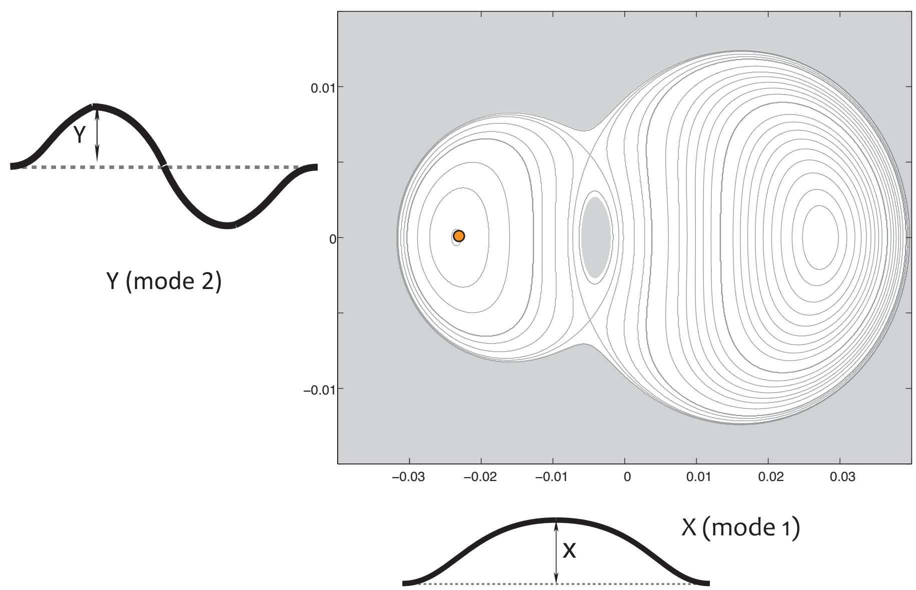
- Examples: chemistry, vehicle dynamics, structural mechanics



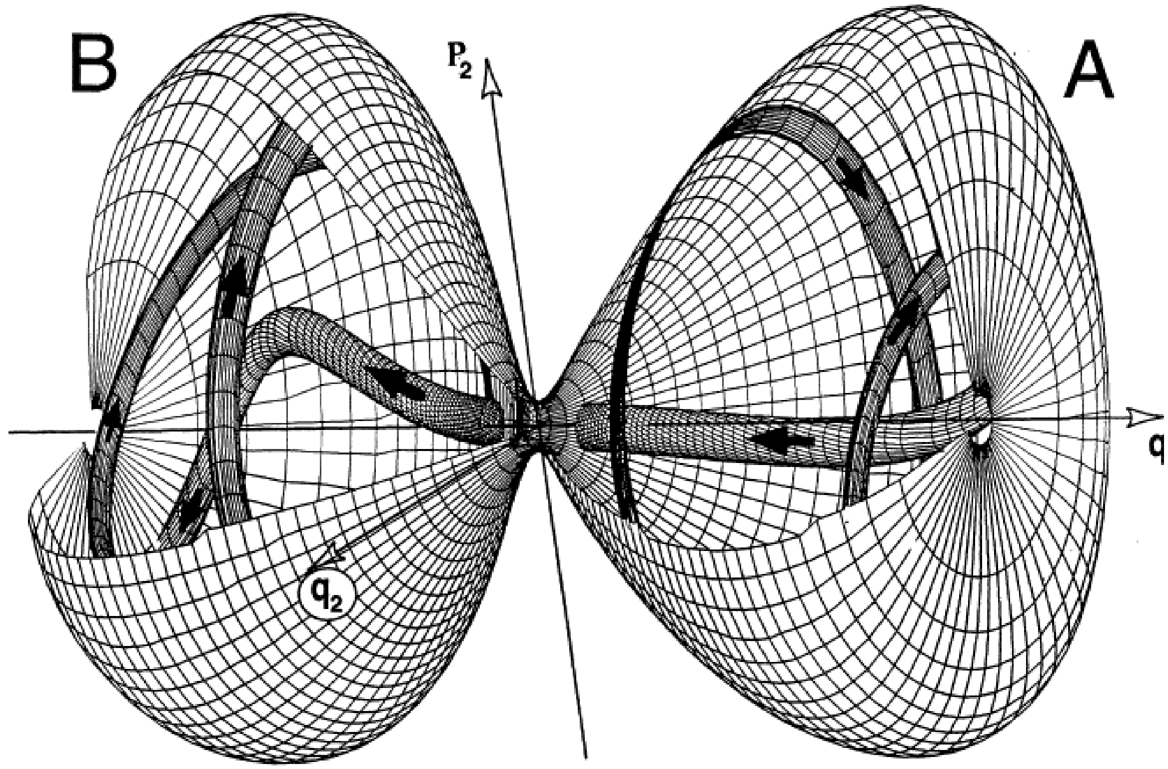








Transitions through bottlenecks via tubes



Topper [1997]

- Wells connected by phase space **transition tubes** $\simeq S^1 \times \mathbb{R}$ for 2 DOF
  — Conley, McGehee, 1960s
  — Llibre, Martínez, Simó, Pollack, Child, 1980s
  — De Leon, Mehta, Topper, Jaffé, Farrelly, Uzer, MacKay, 1990s
  — Gómez, Koon, Lo, Marsden, Masdemont, Ross, Yanao, 2000s

Motion near index 1 saddles

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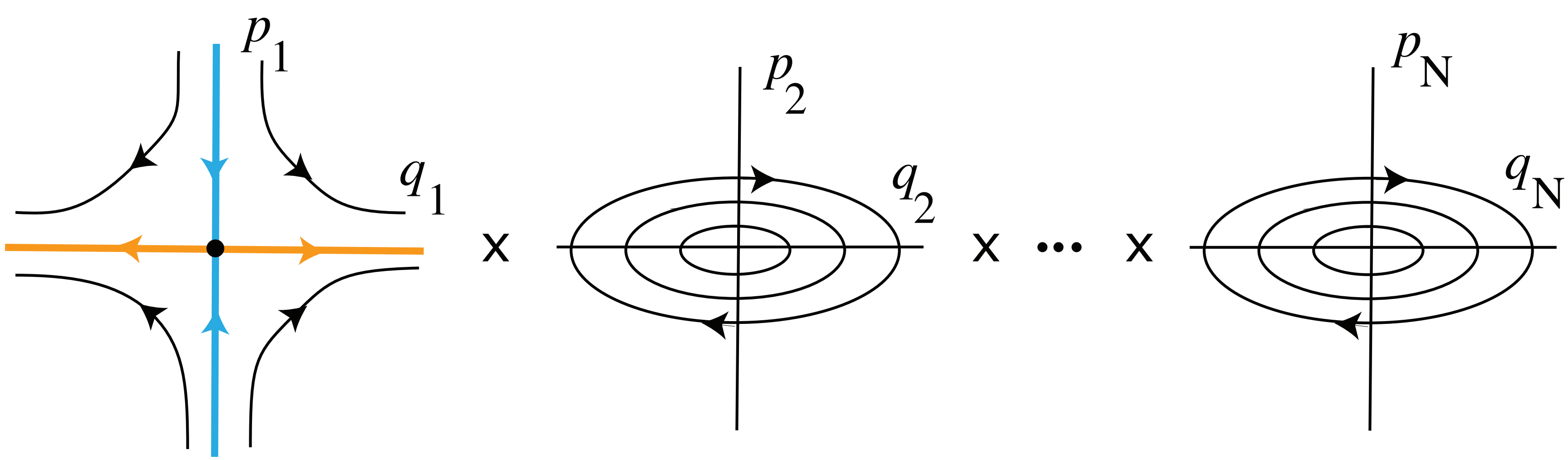
- **Bottleneck region** is a saddle $\times$ center $\times \cdots \times$ center ($N-1$ centers)



the saddle-space projection and $N-1$ center projections — the N canonical planes

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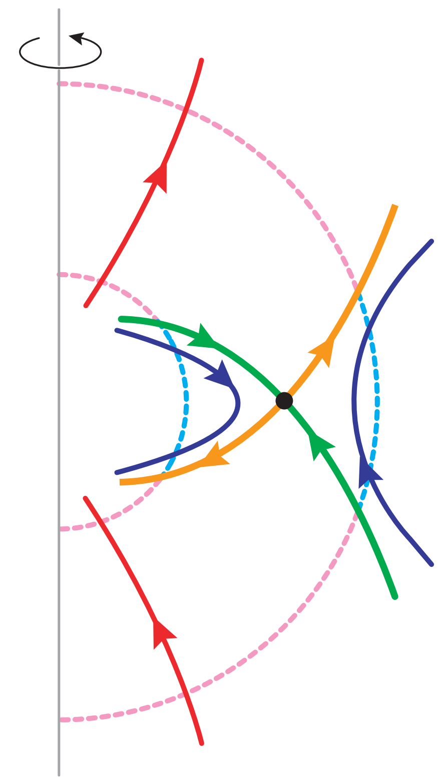
so $\mathcal{M}_{\Delta E} \simeq S^1$ is just a periodic orbit of period $T_{\text{po}} = 2\pi/\omega$

McGehee representation of energy surface

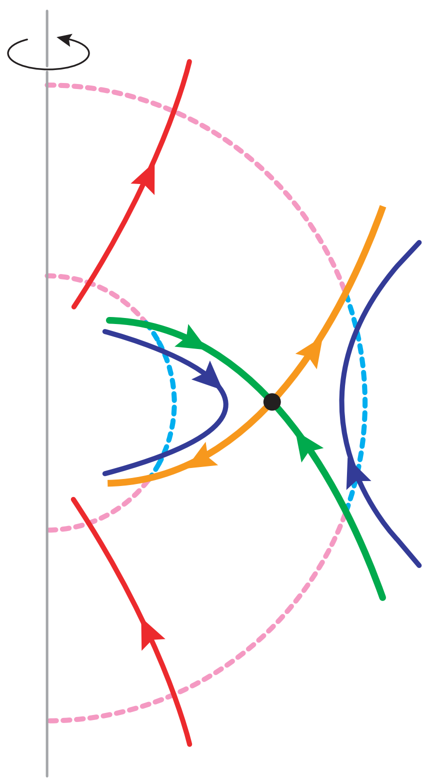
- Cylindrical **tubes** of trajectories asymptotic to $\mathcal{M}_{\Delta E}$: stable & unstable invariant manifolds, $W^s_{\pm}(\mathcal{M}_{\Delta E}), W^u_{\pm}(\mathcal{M}_{\Delta E})$, $\simeq S^1 \times \mathbb{R}$
- Tubes enclose transitioning trajectories crossing the bottleneck

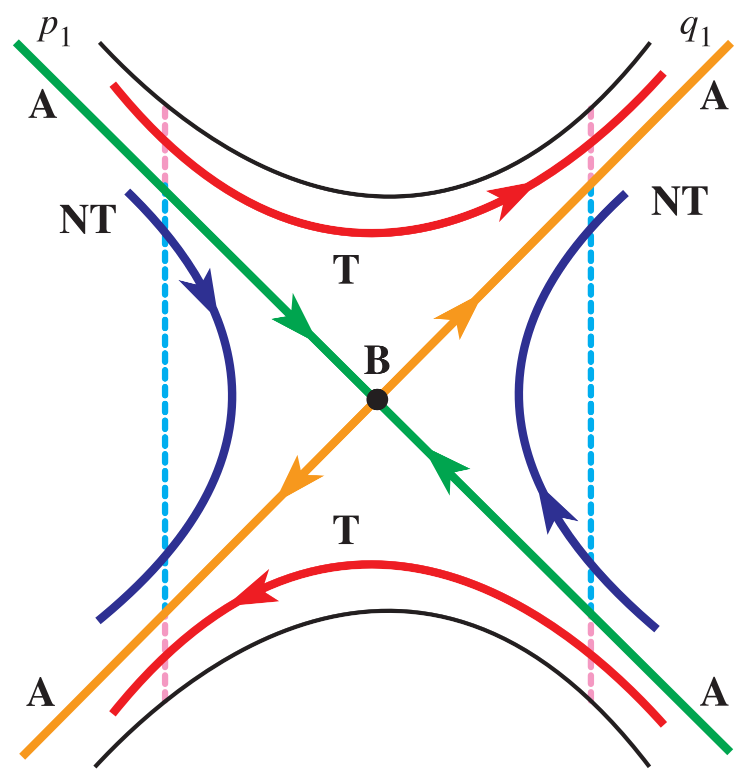


McGehee representation of energy surface of the structure $S^2 \times \mathbb{R}$

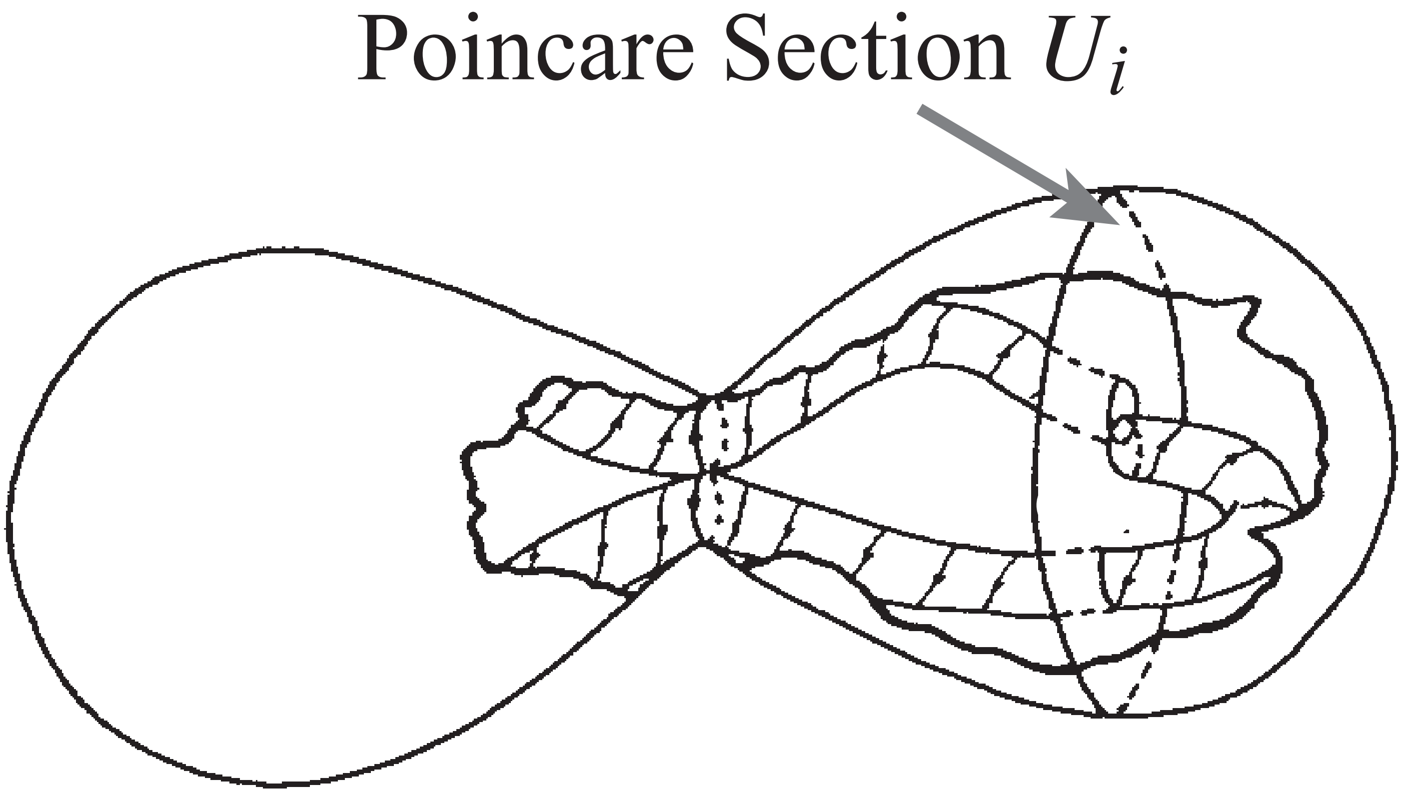
McGehee representation of energy surface

- **B** : **bounded orbits** (periodic)
- **A** : asymptotic **stable** and **unstable** manifolds to B (tubes)
- **T** : **transitioning** and **NT** : **non-transitioning** trajectories





Tube dynamics — global picture



De Leon [1992]

- **Tube dynamics:** All transitioning motion between wells connected by bottlenecks must occur through tube
  - Imminent transition regions, transitioning fractions
  - Consider k Poincaré sections U_i, various excess energies ΔE

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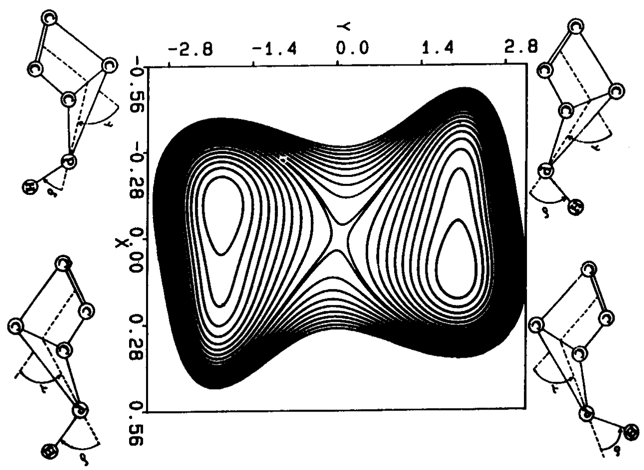

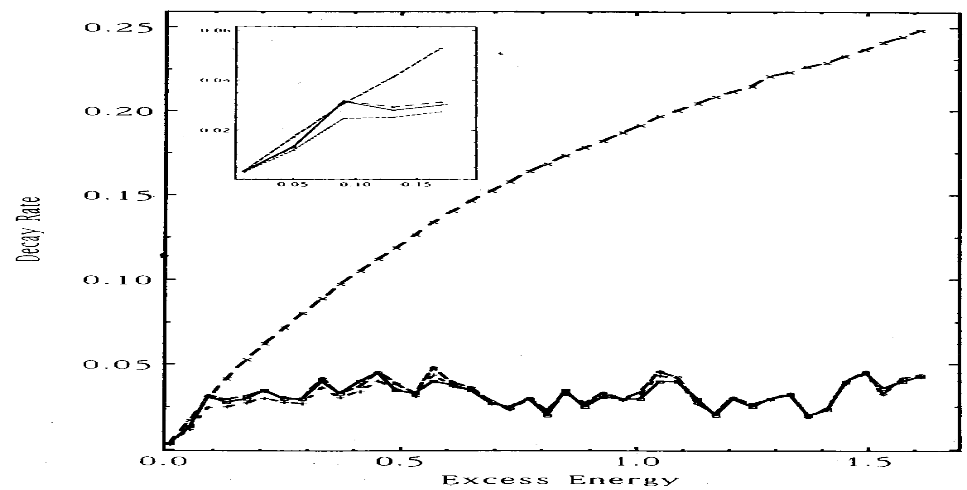

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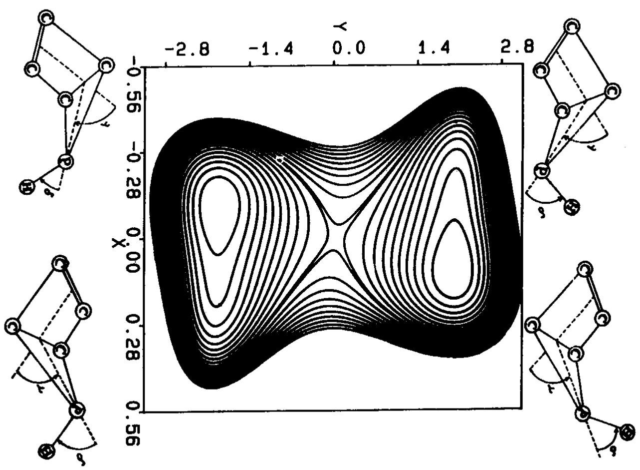
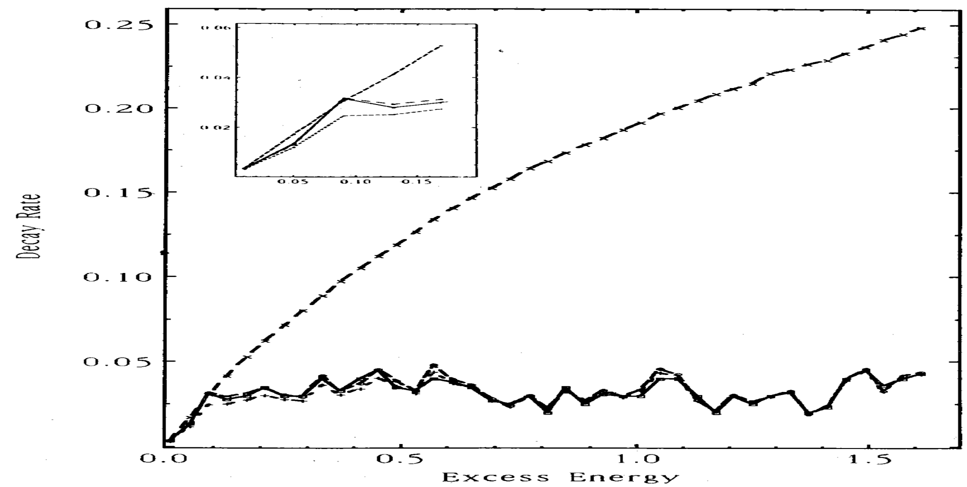
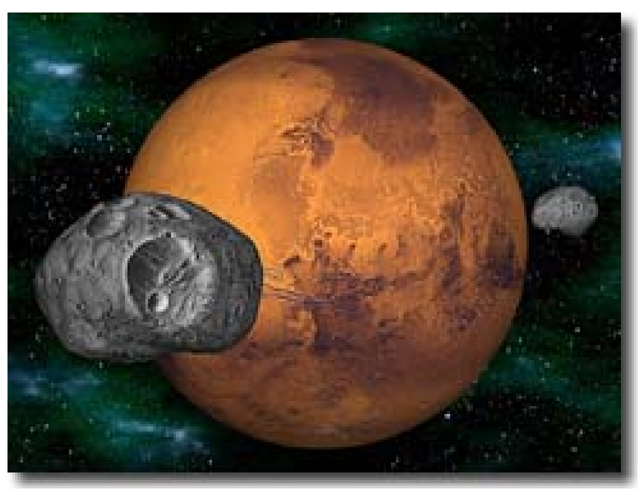
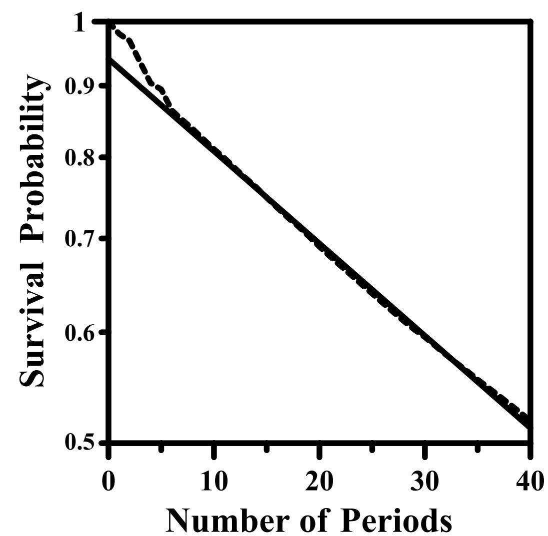




  - — celestial mechanics, asteroid escape rates e.g., Jaffé, Ross, Lo, Marsden, Farrelly, Uzer [2002]





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- **Preferentially triggering or avoiding transitions**
— ship stability / capsize, etc.

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- **Structural mechanics**
— re-configurable deformation of flexible objects

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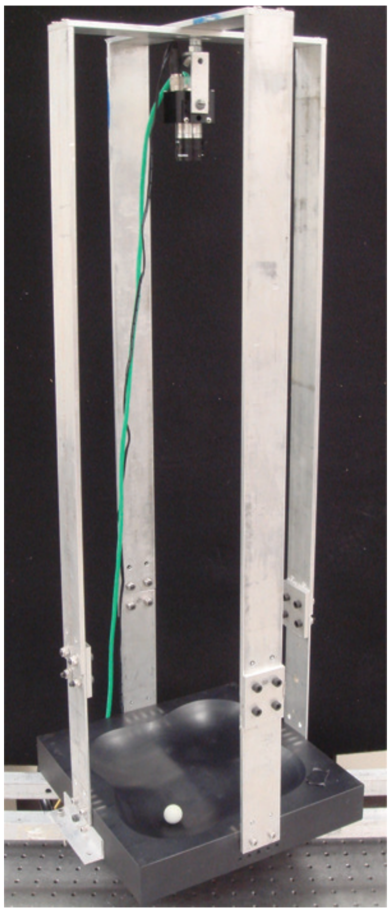
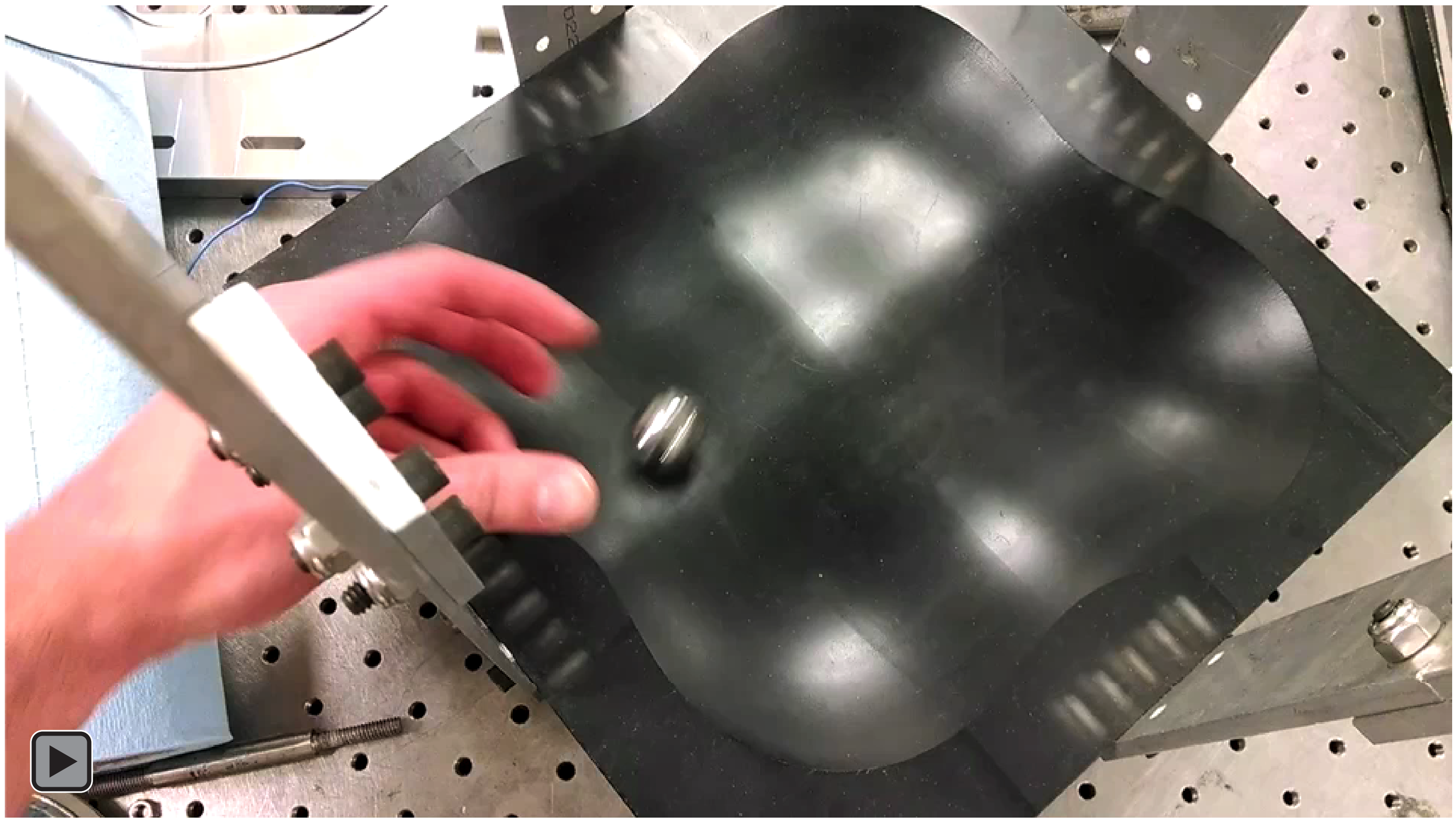


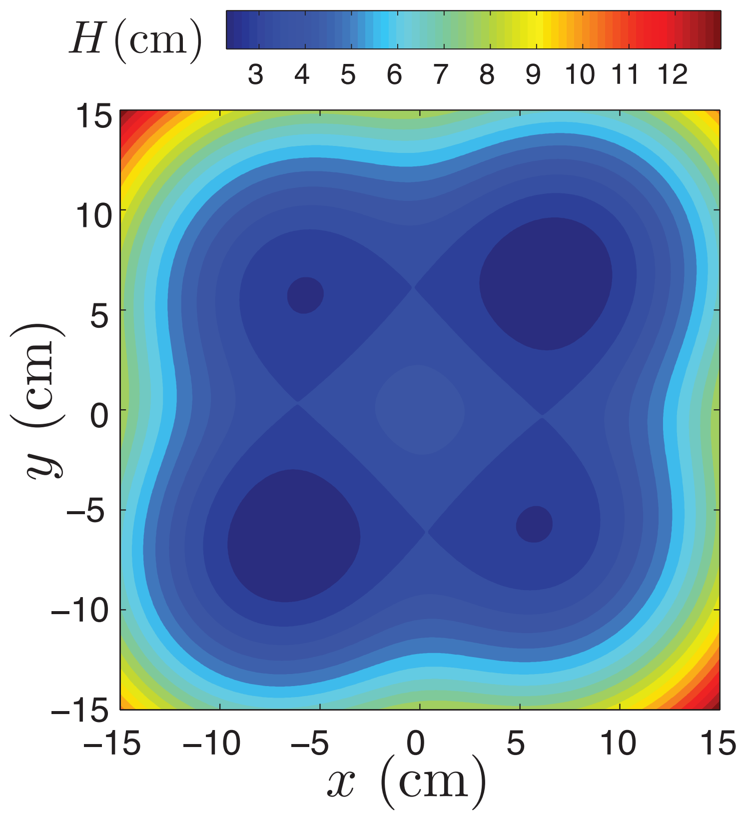


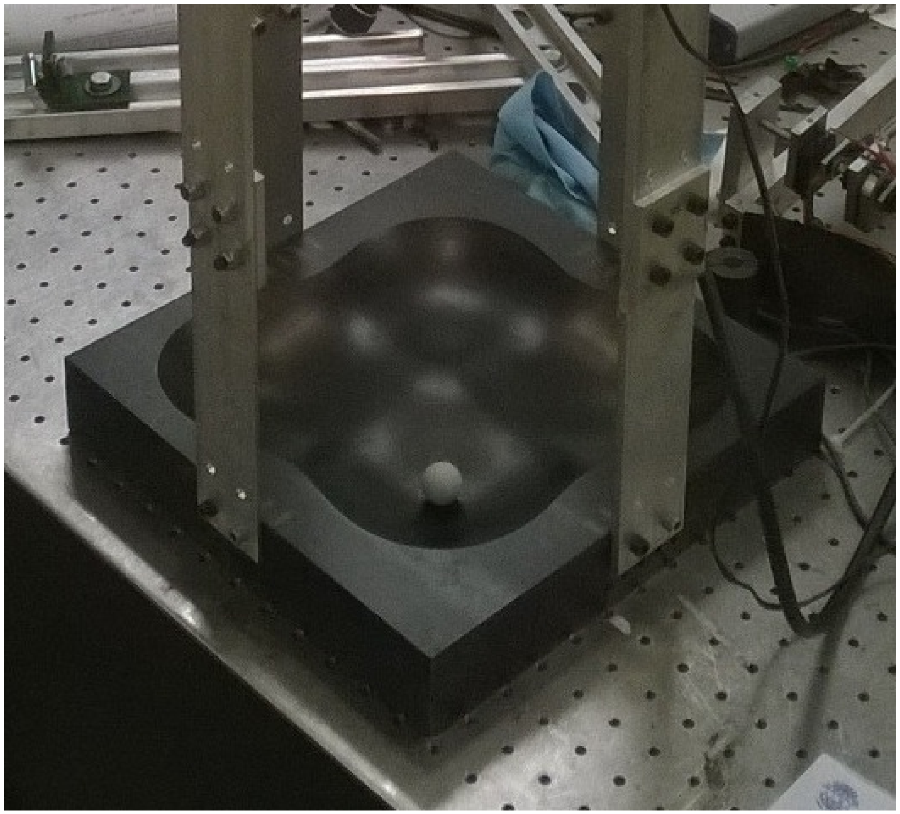
Virgin, Lyman, Davis [2010] Am. J. Phys.

Ball rolling on a surface — 2 DOF

- The potential energy is $V(x,y) = gH(x,y)$, where the surface is arbitrary, e.g., we chose

$$H(x,y) = \alpha(x^2+y^2) - \beta(\sqrt{x^2+\gamma} + \sqrt{y^2+\gamma}) - \xi xy + H_0.$$





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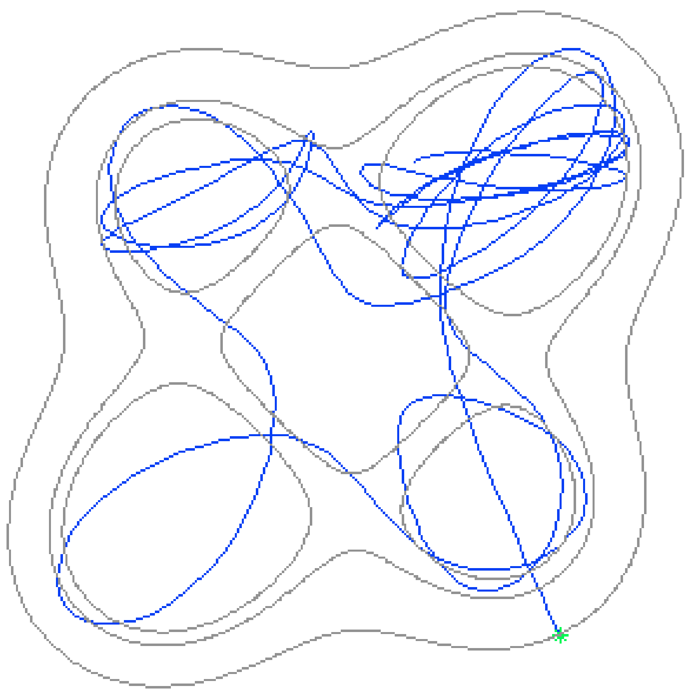
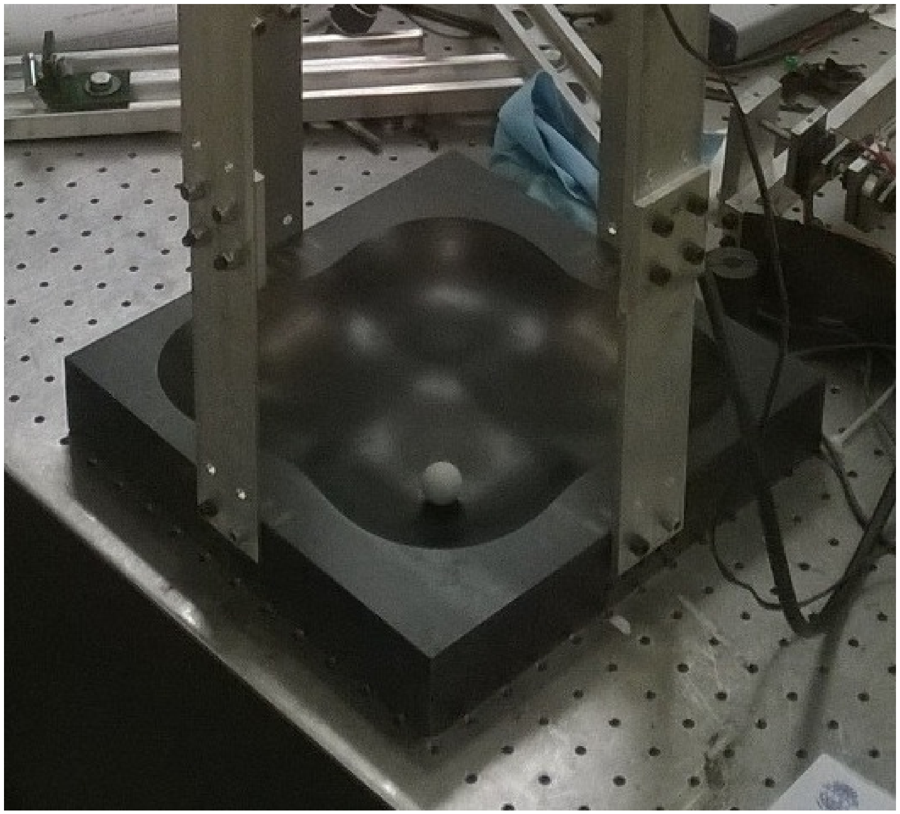
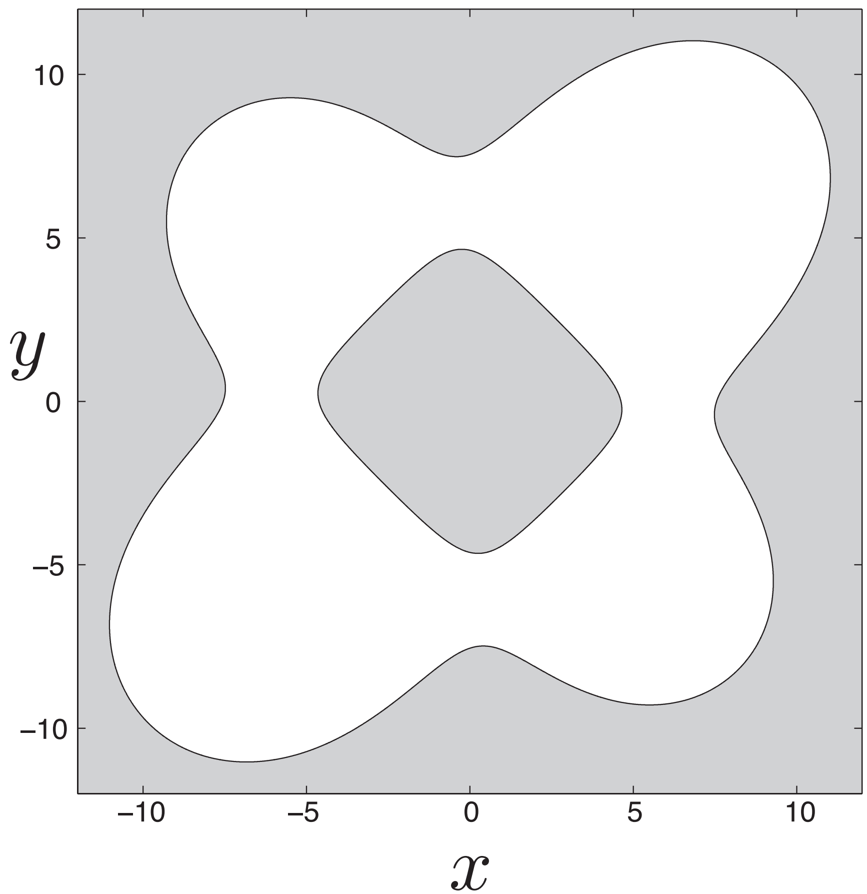




typical experimental trial

Transition tubes in the rolling ball system



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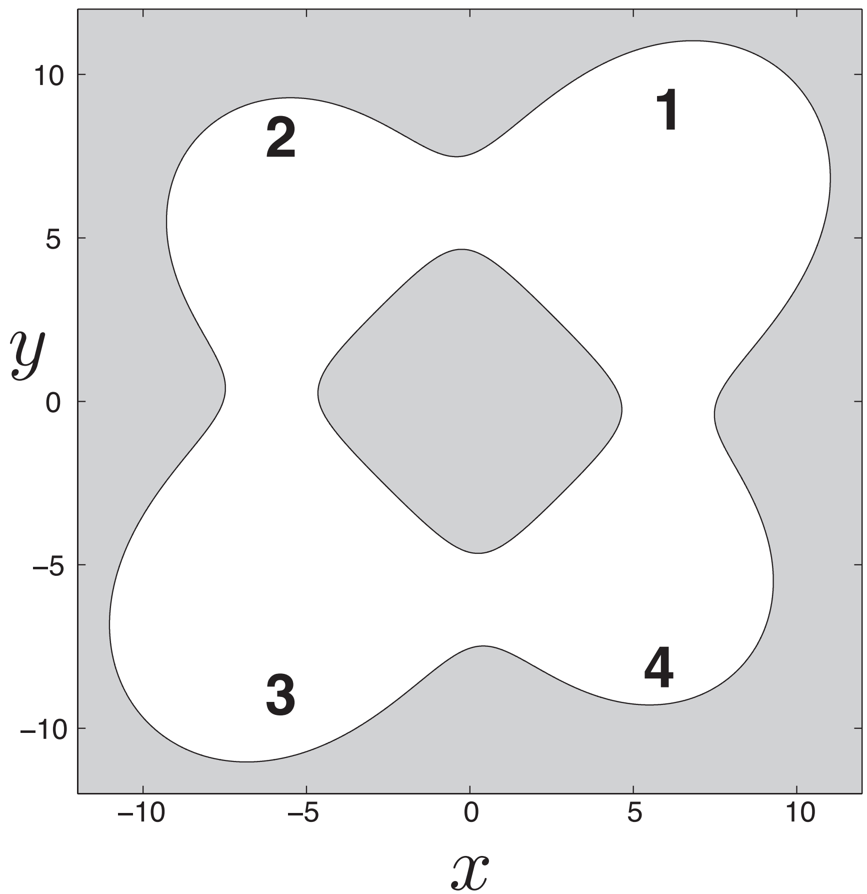

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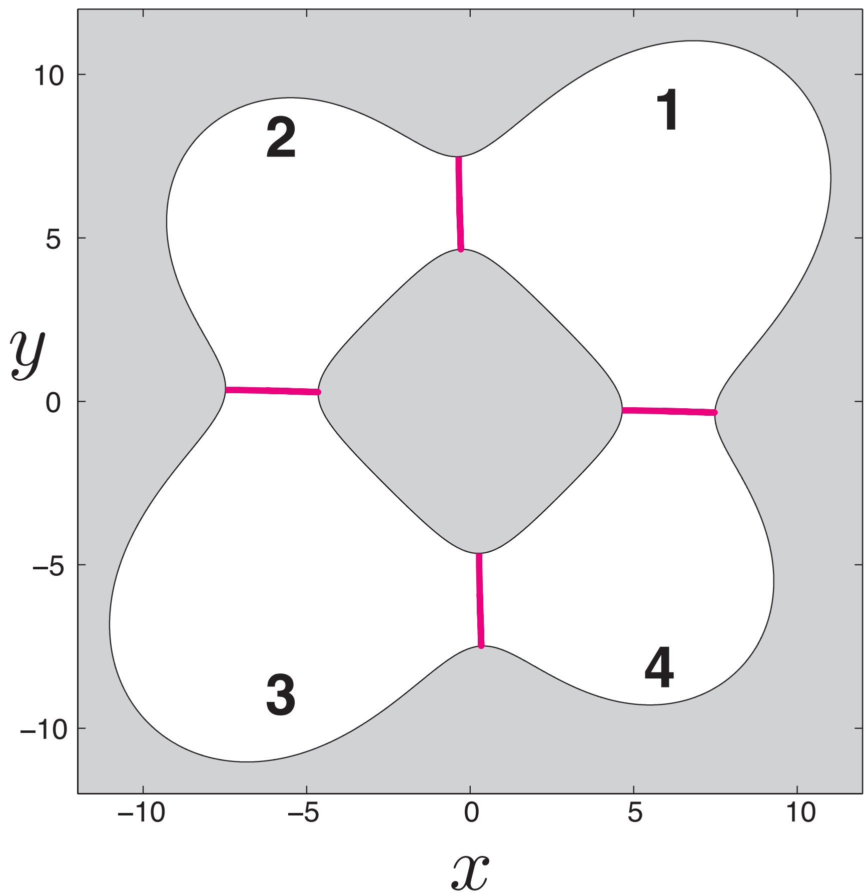

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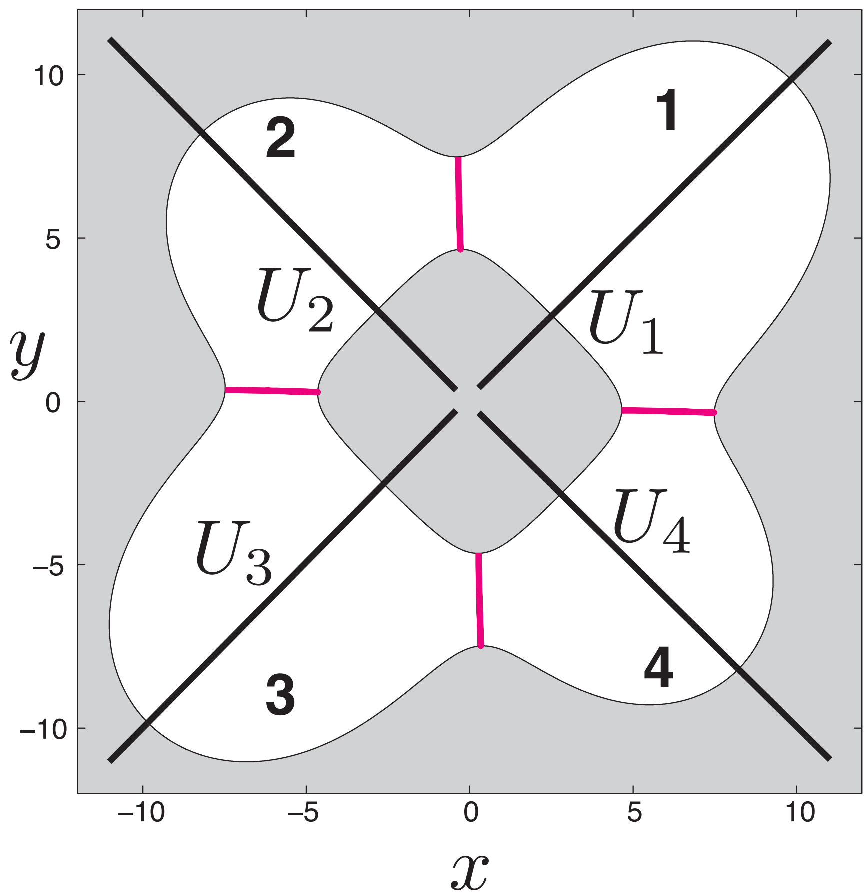

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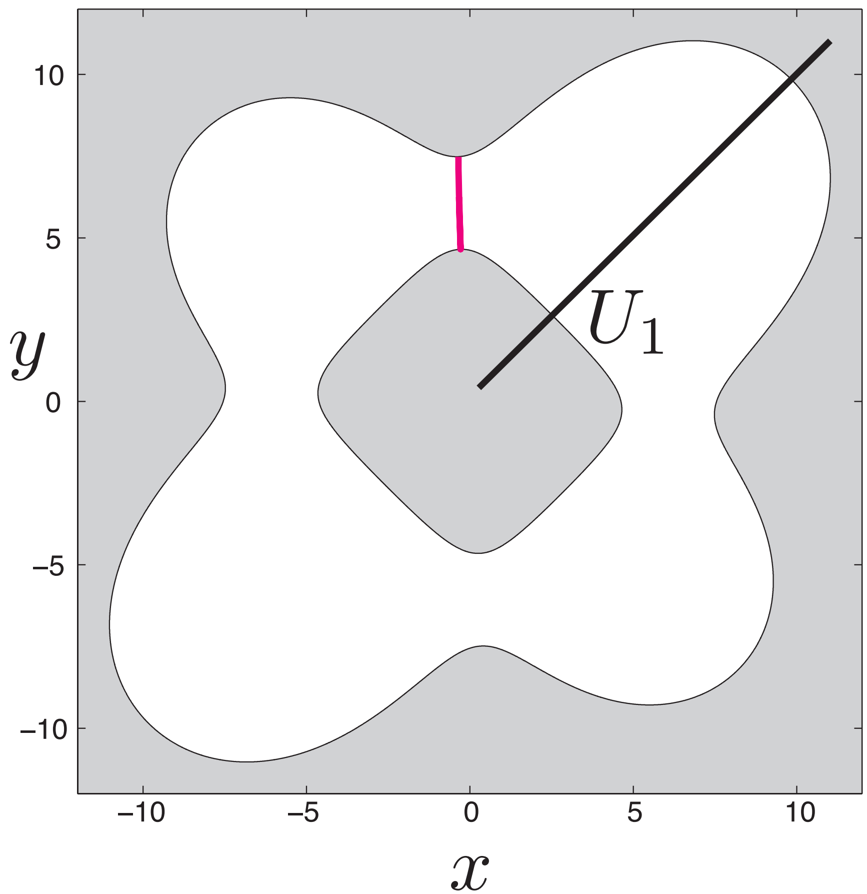

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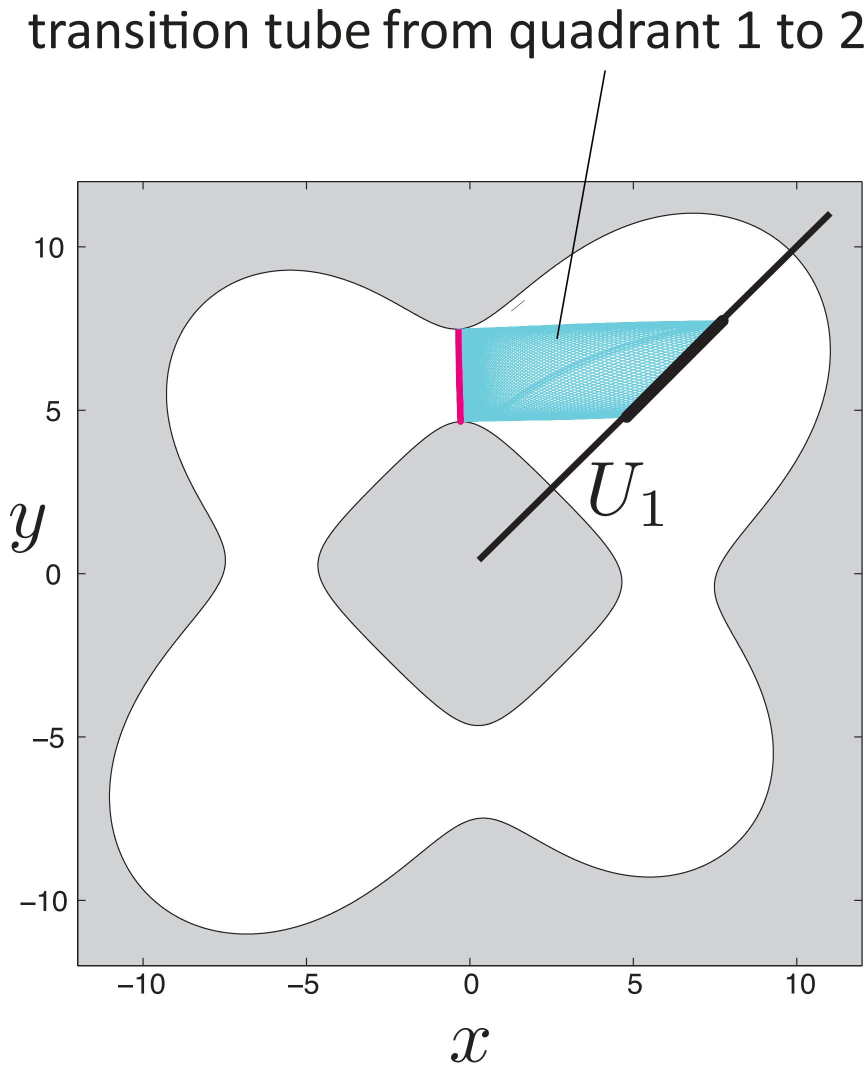

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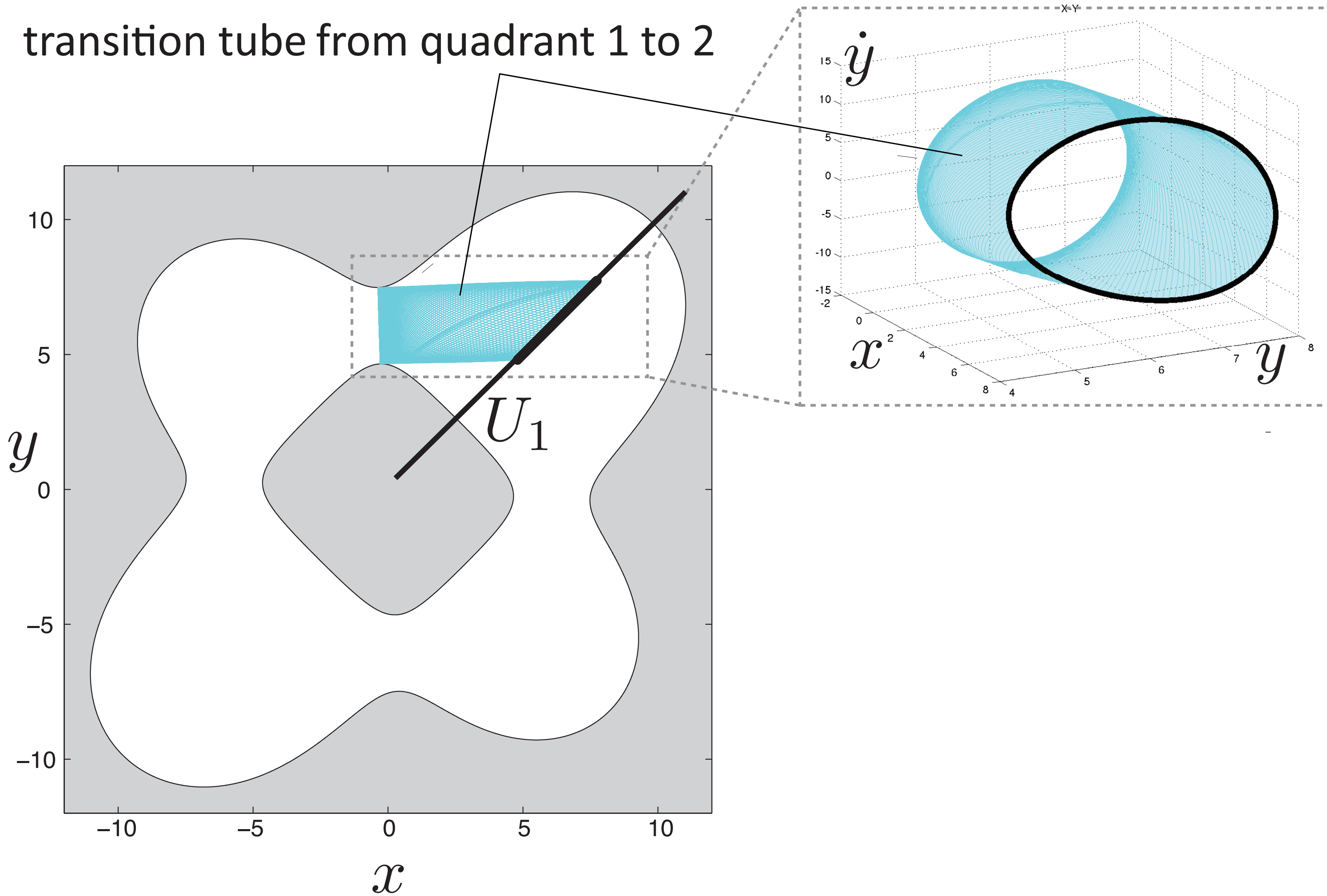

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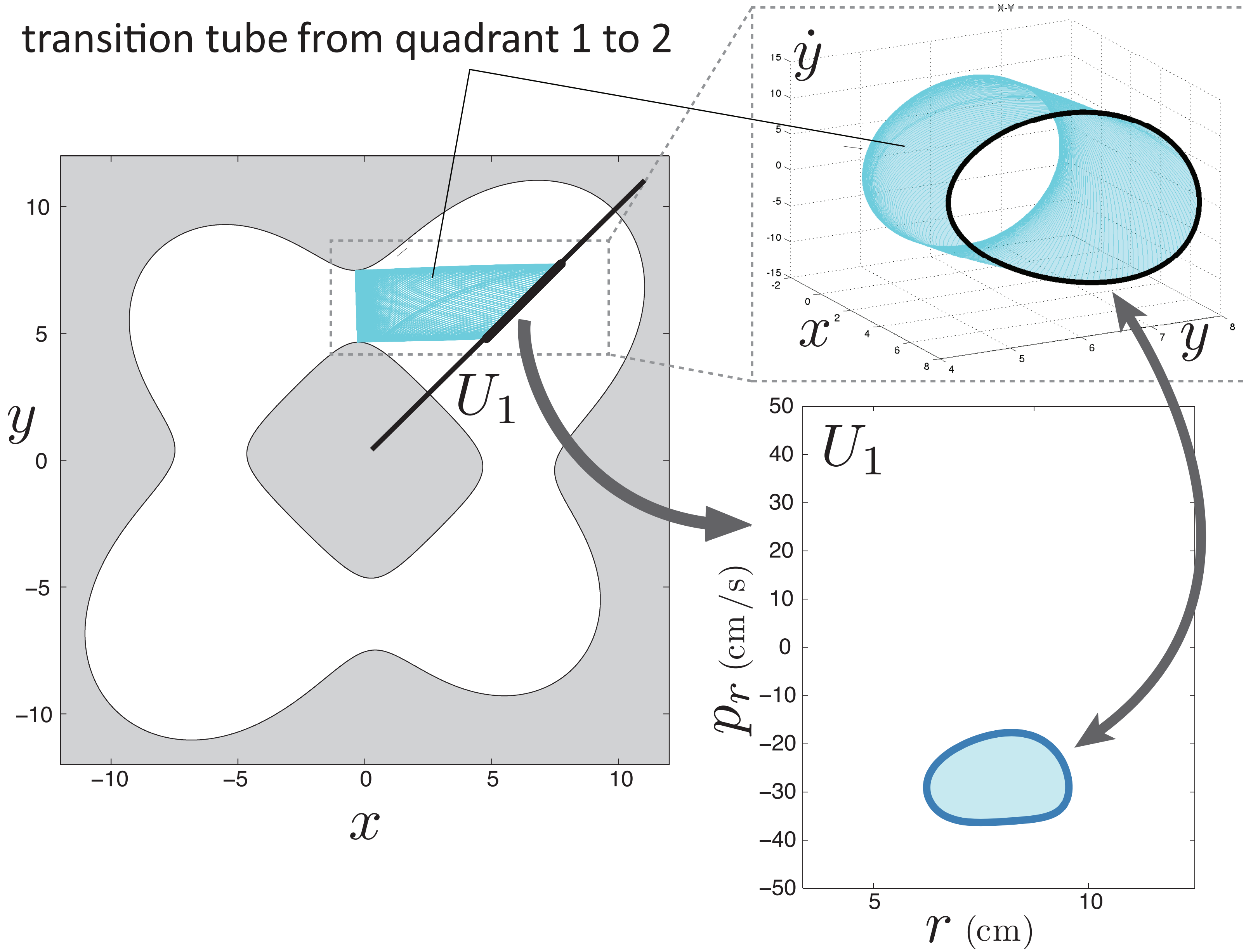

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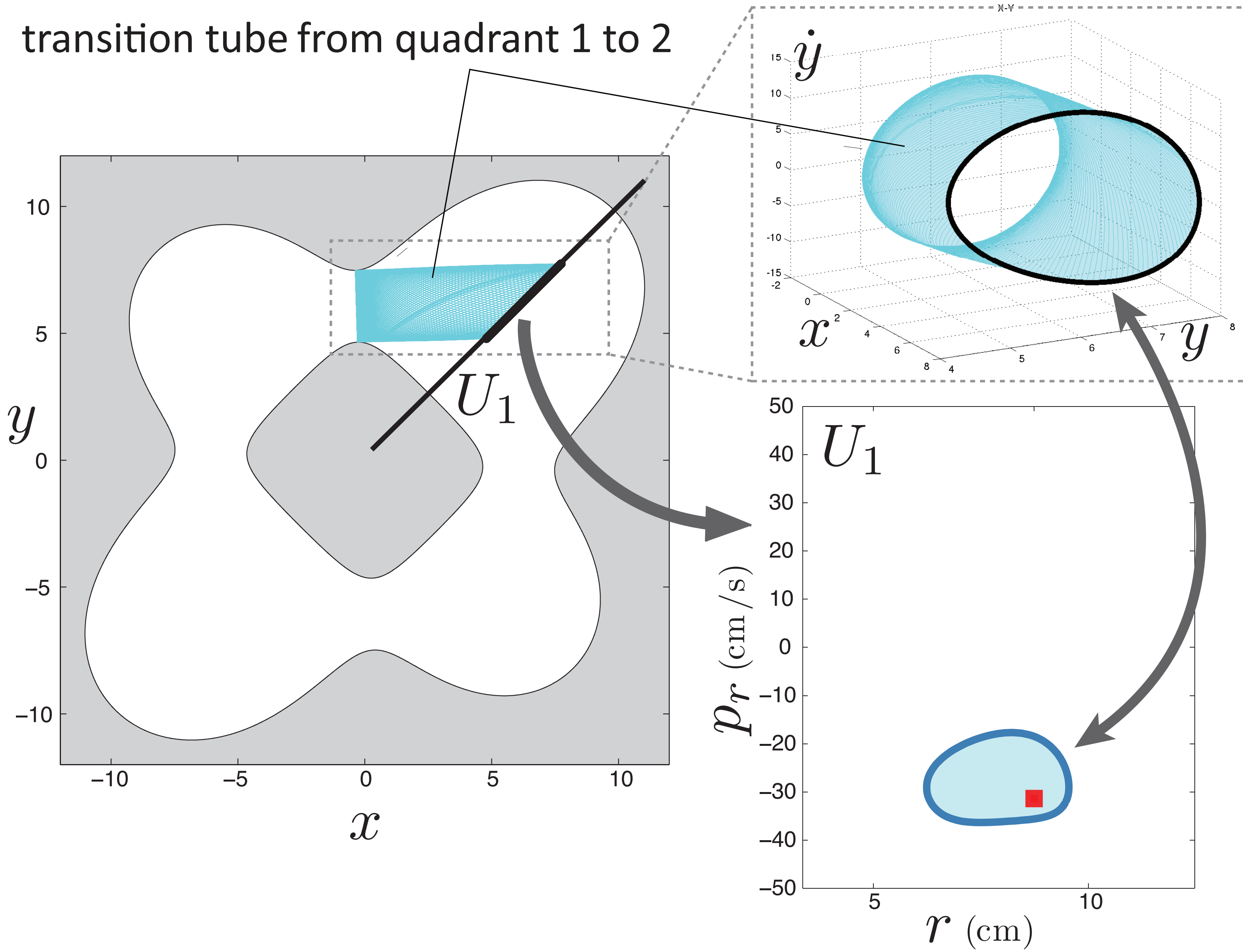

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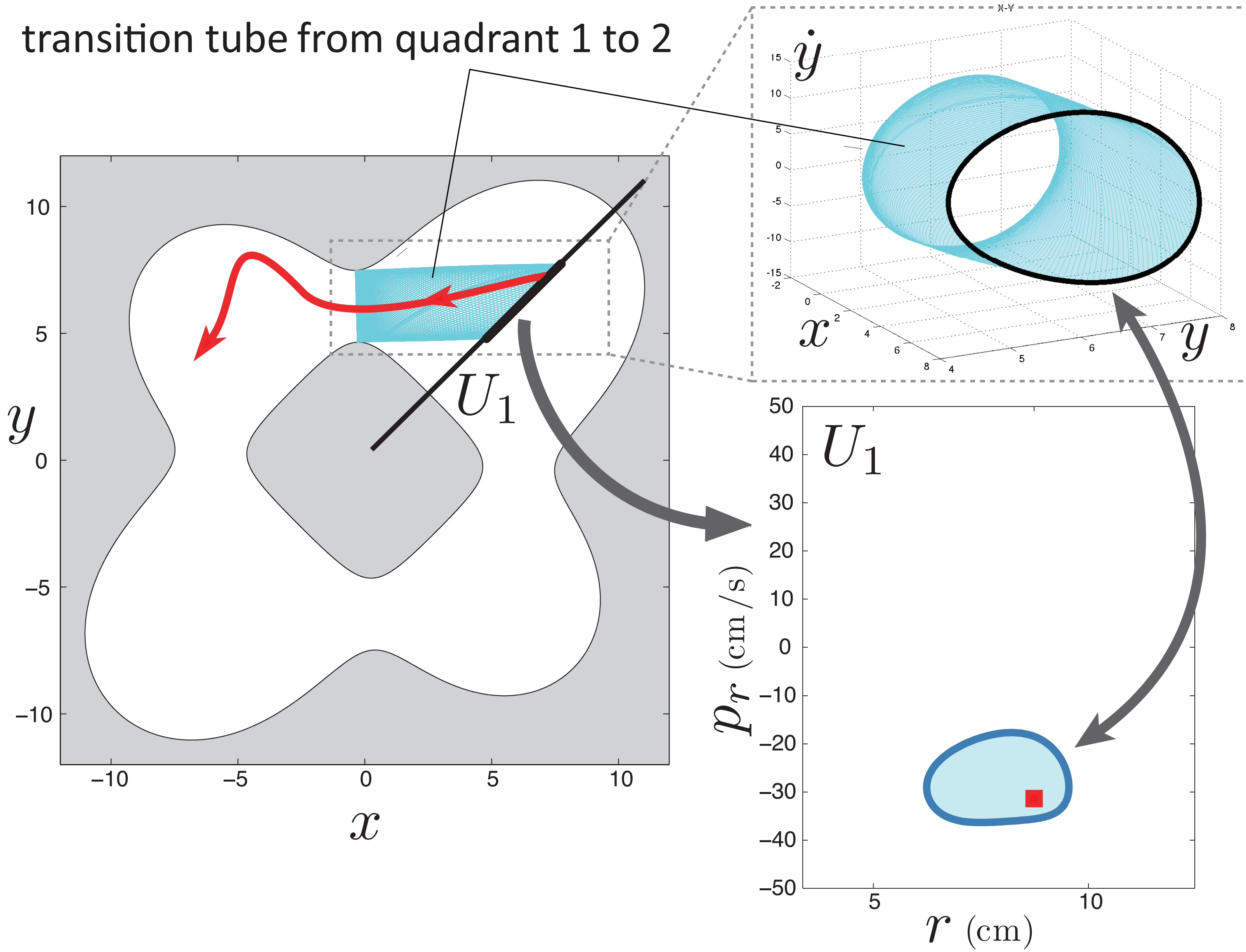

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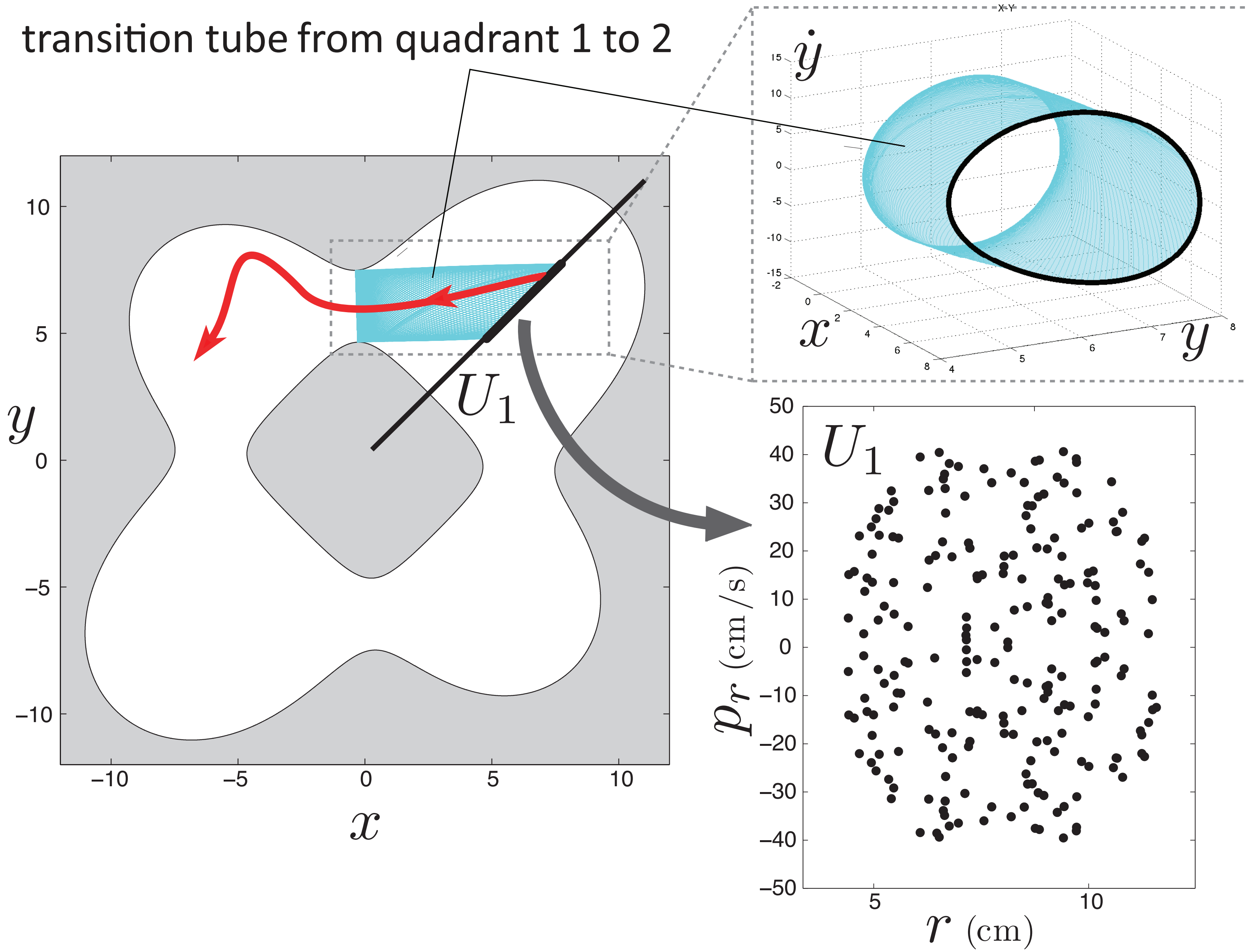

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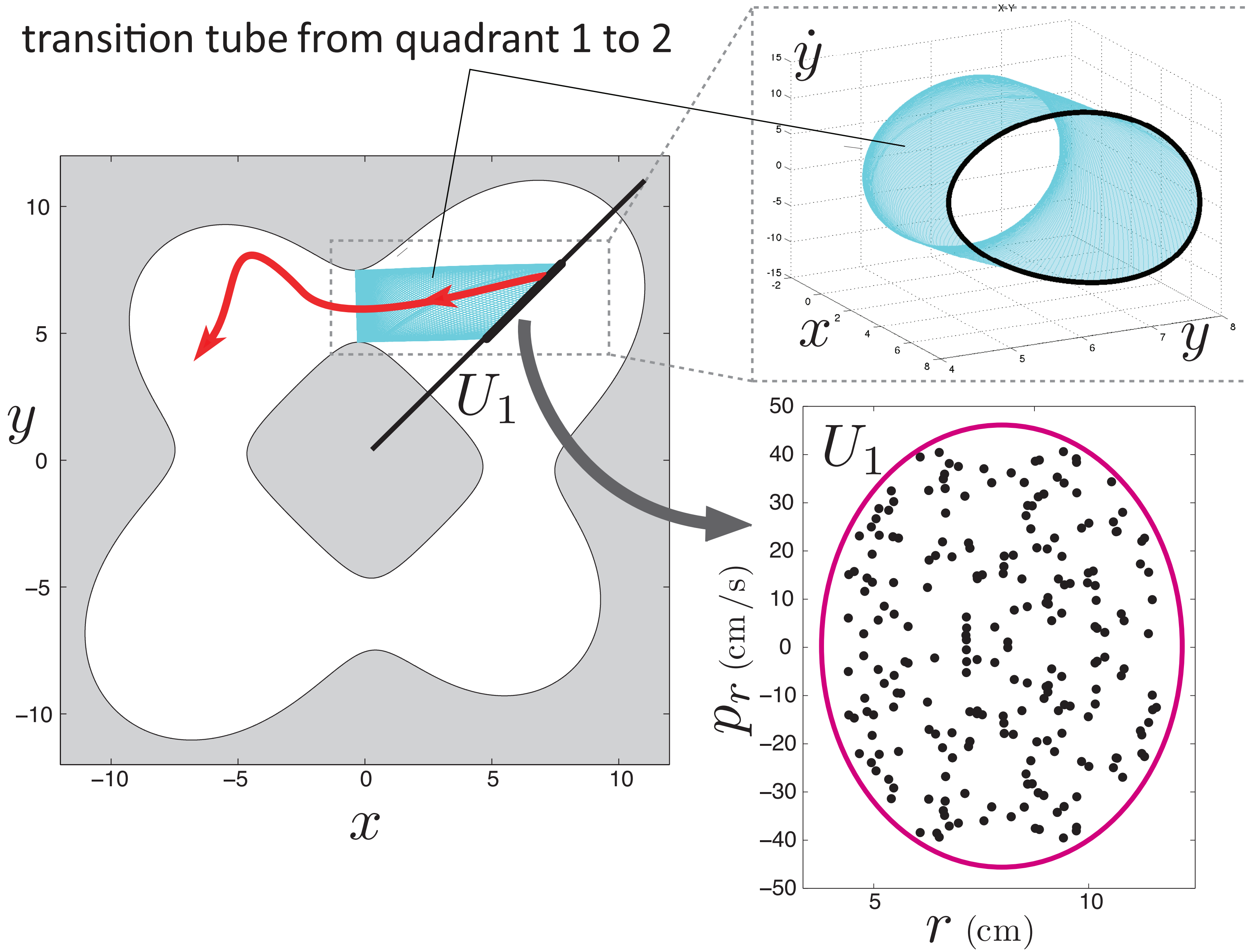

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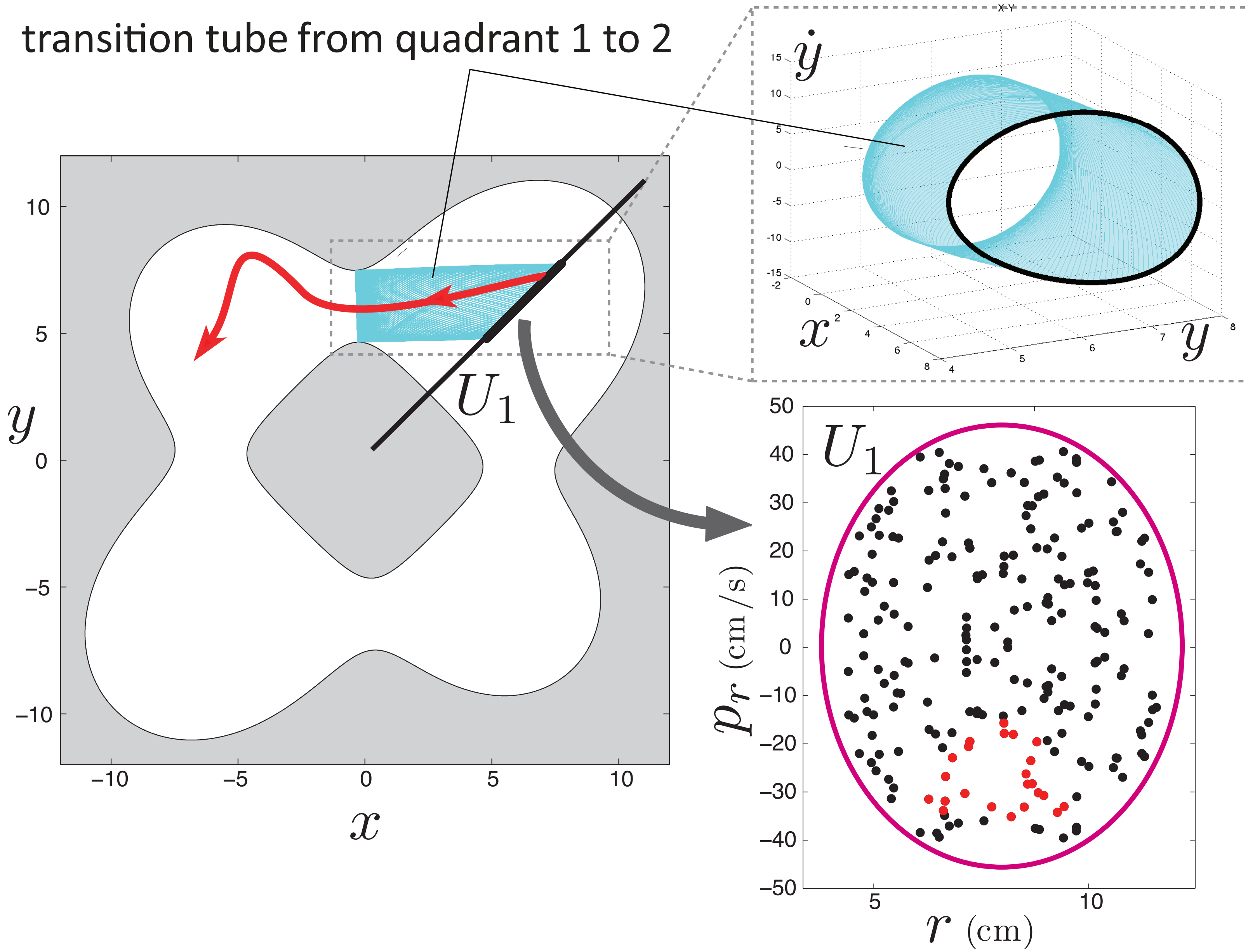

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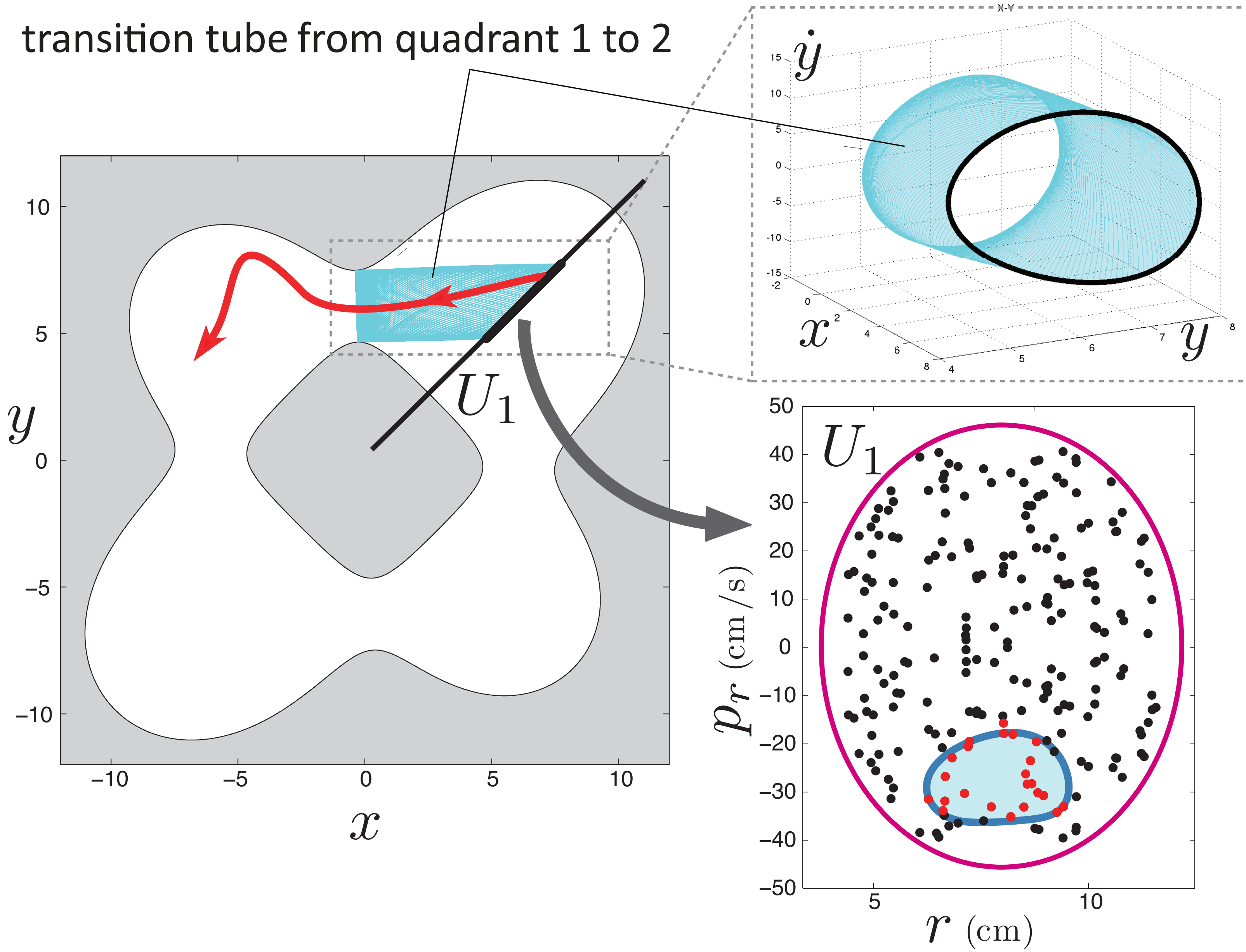
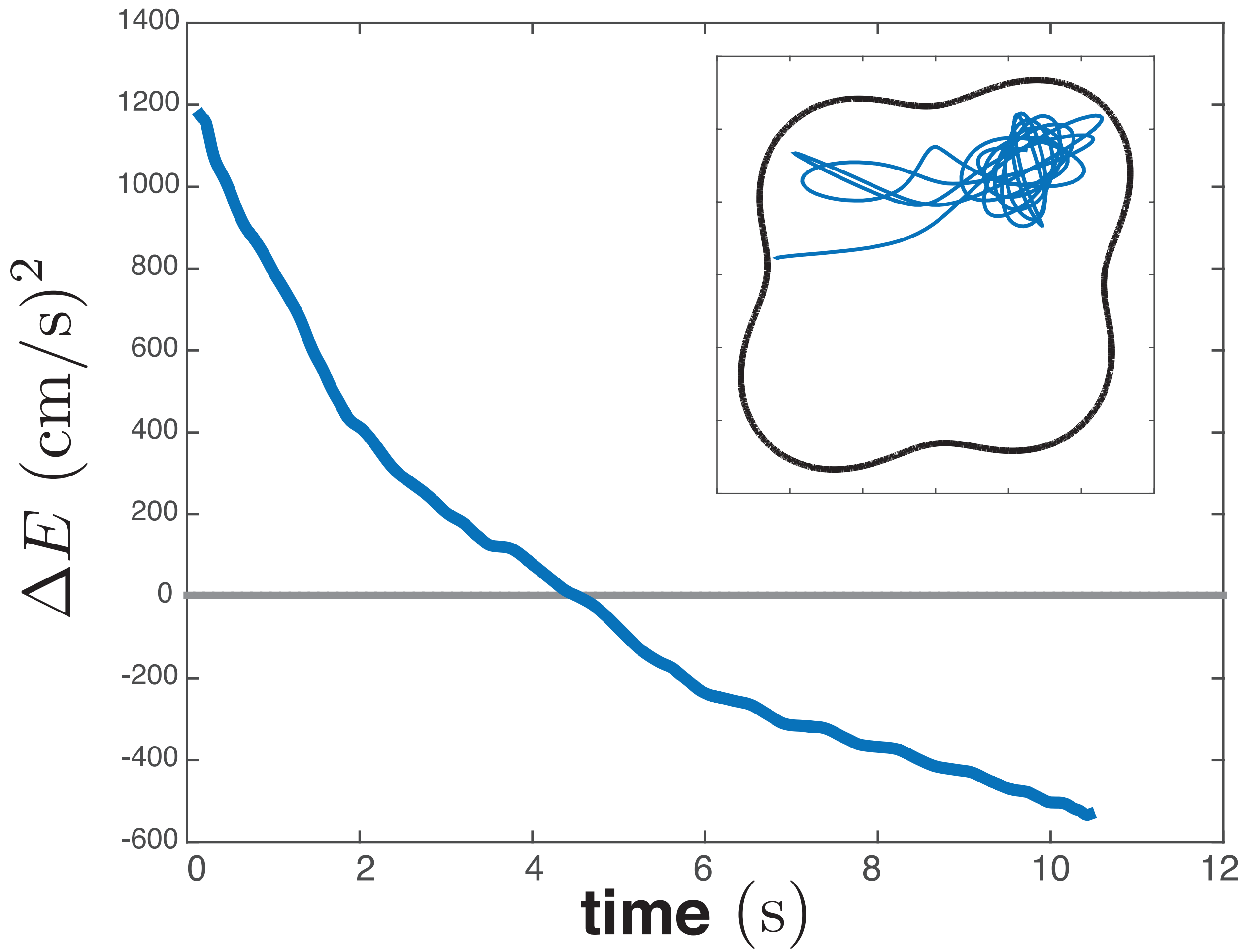


Analysis of experimental data



- 120 experimental trials of about 10 seconds each, recorded at 50 Hz

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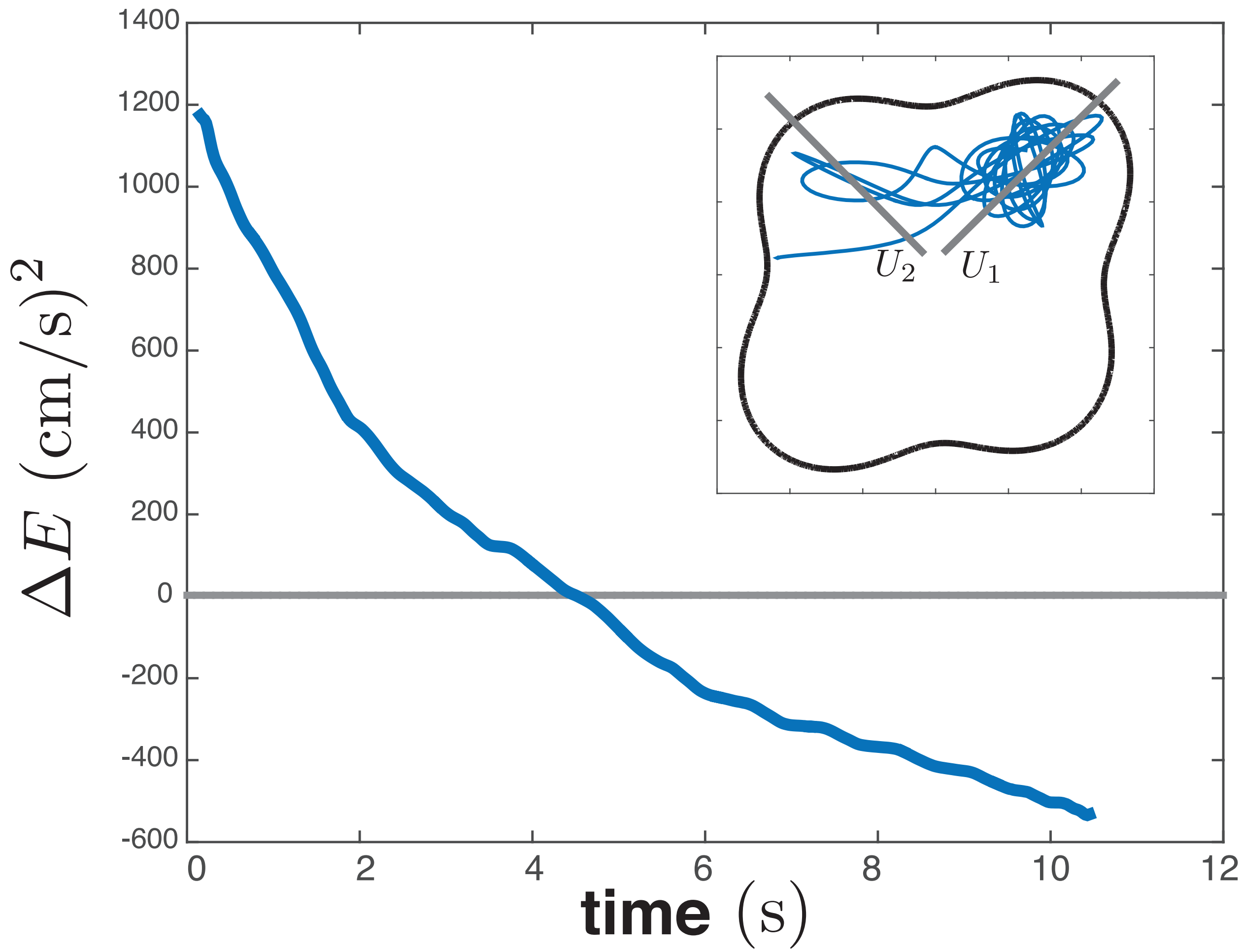

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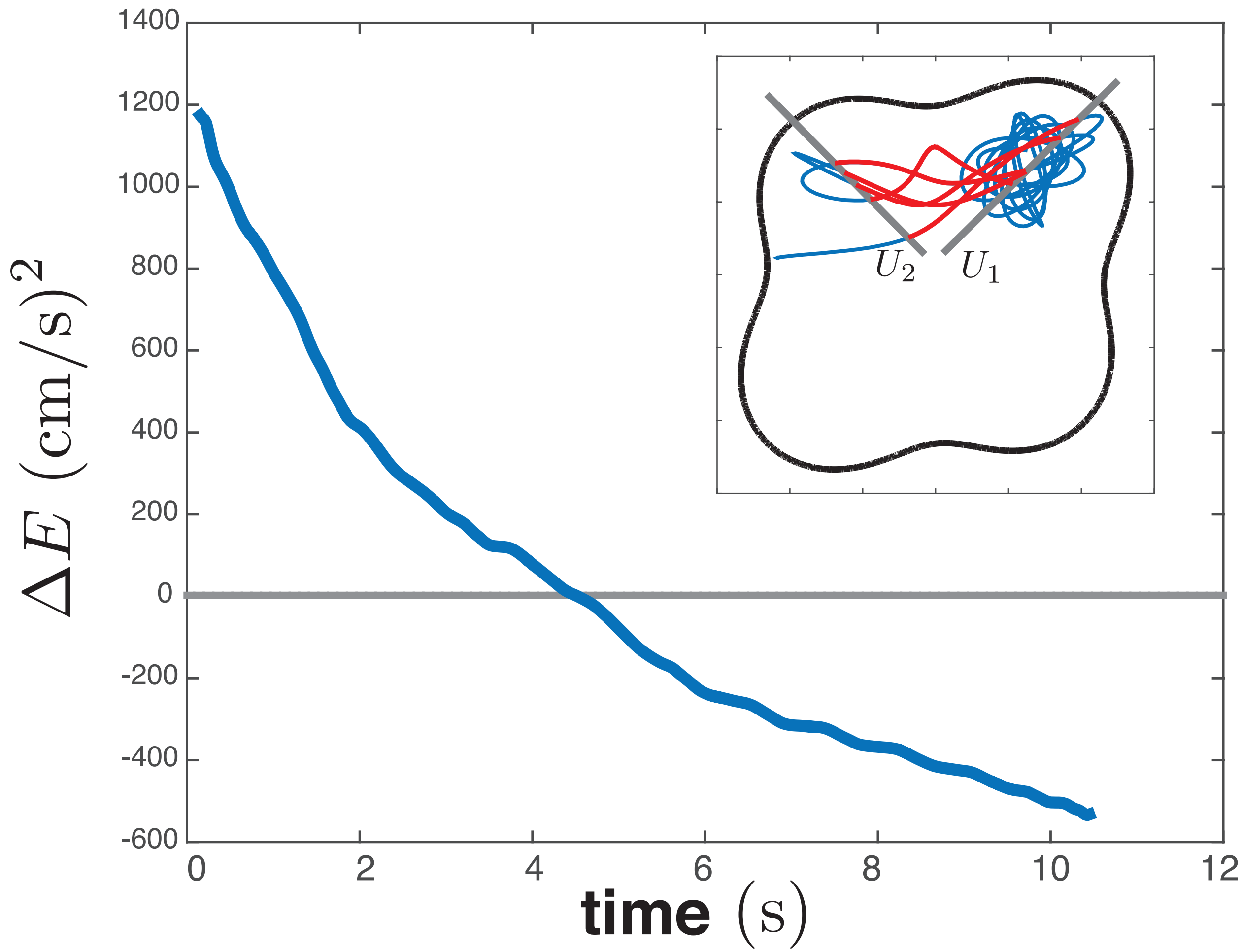

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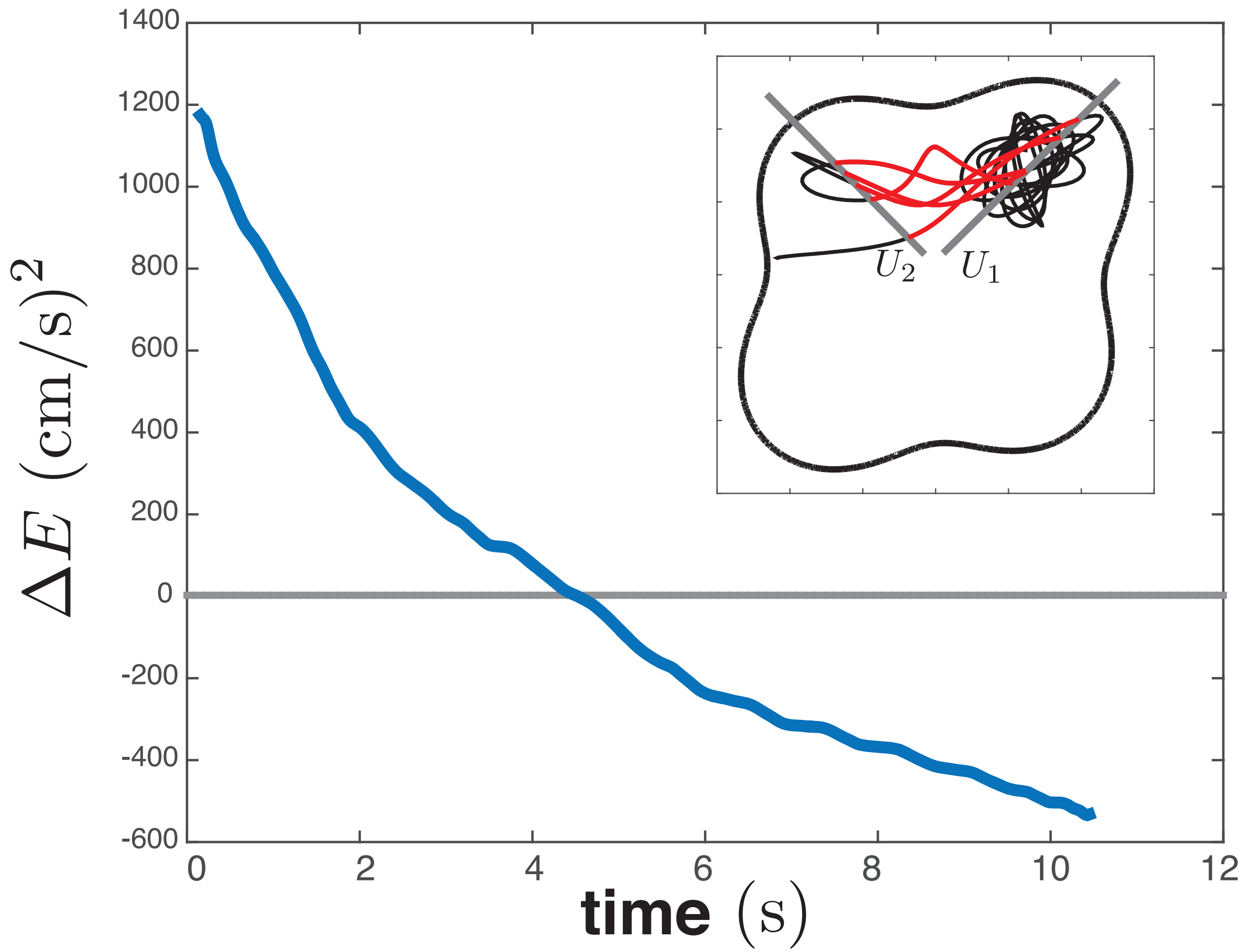

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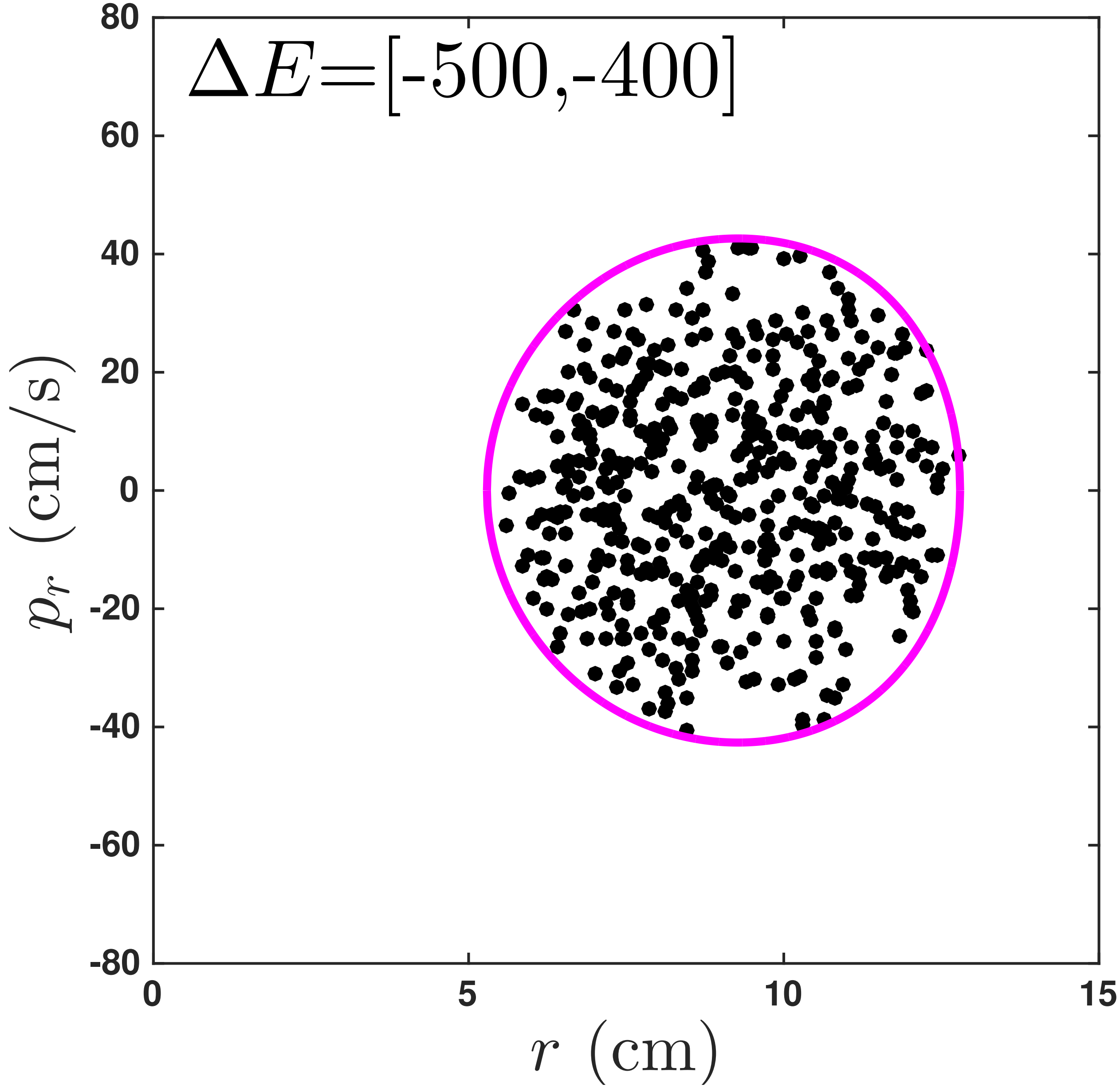

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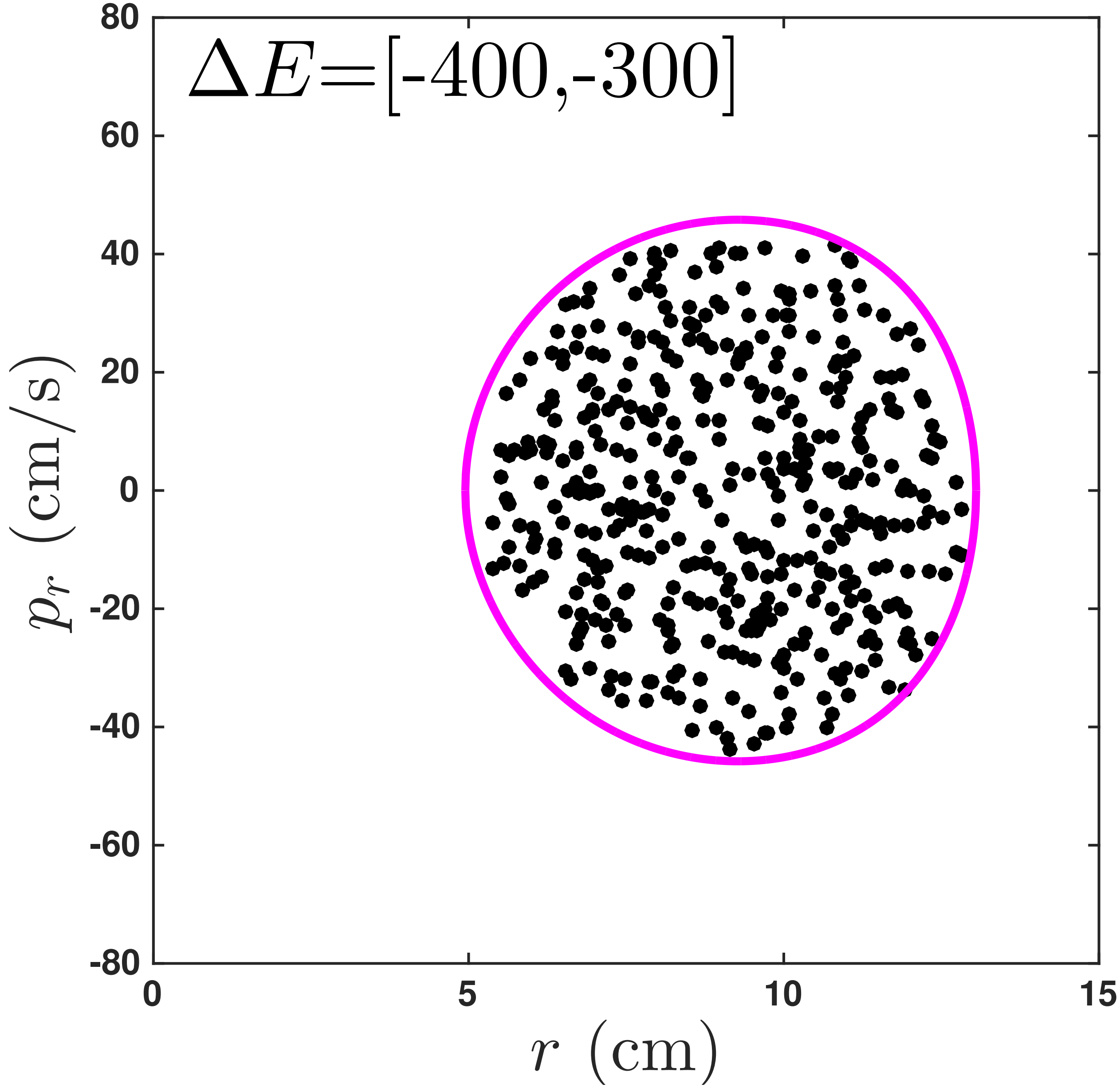

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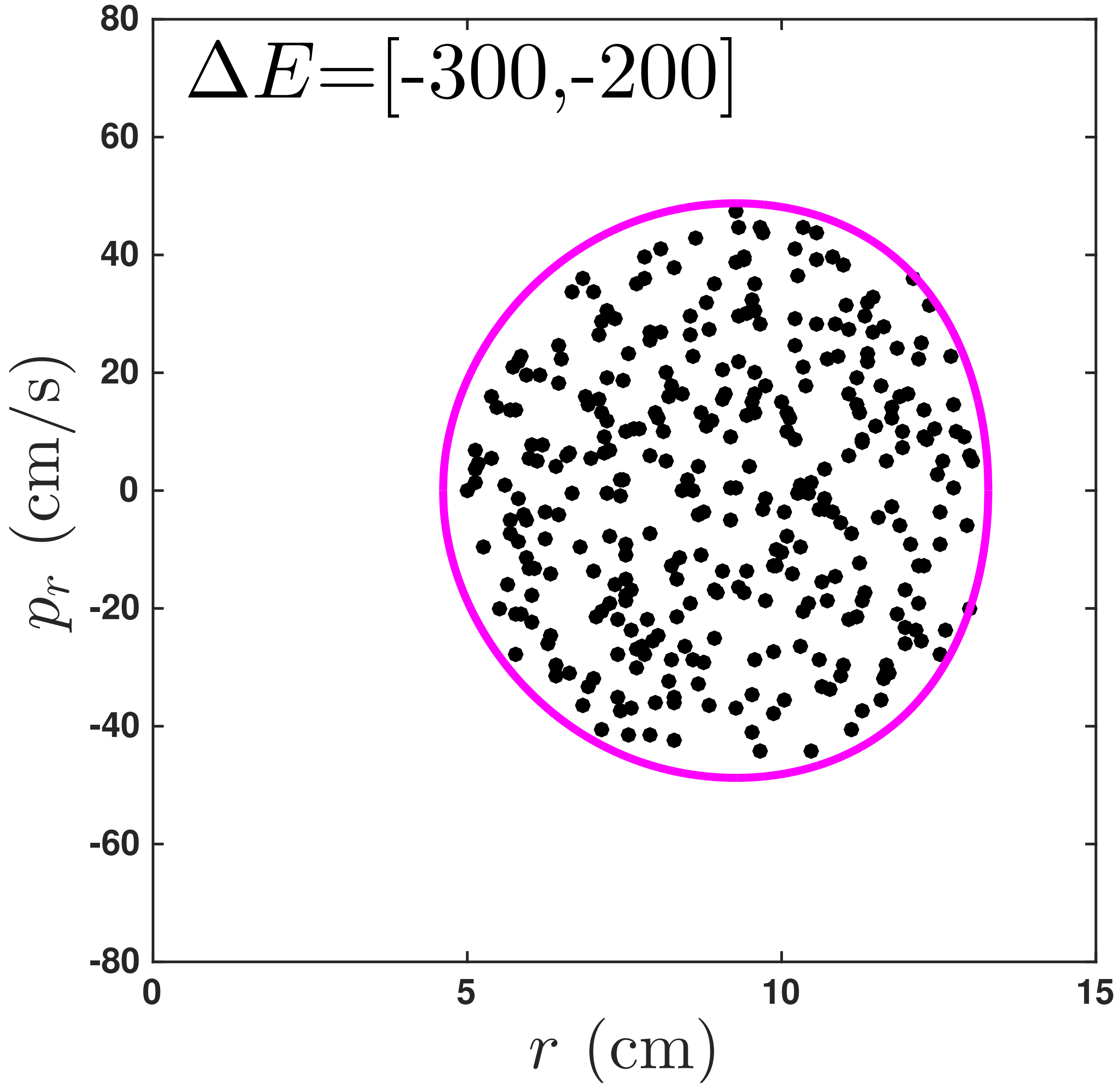

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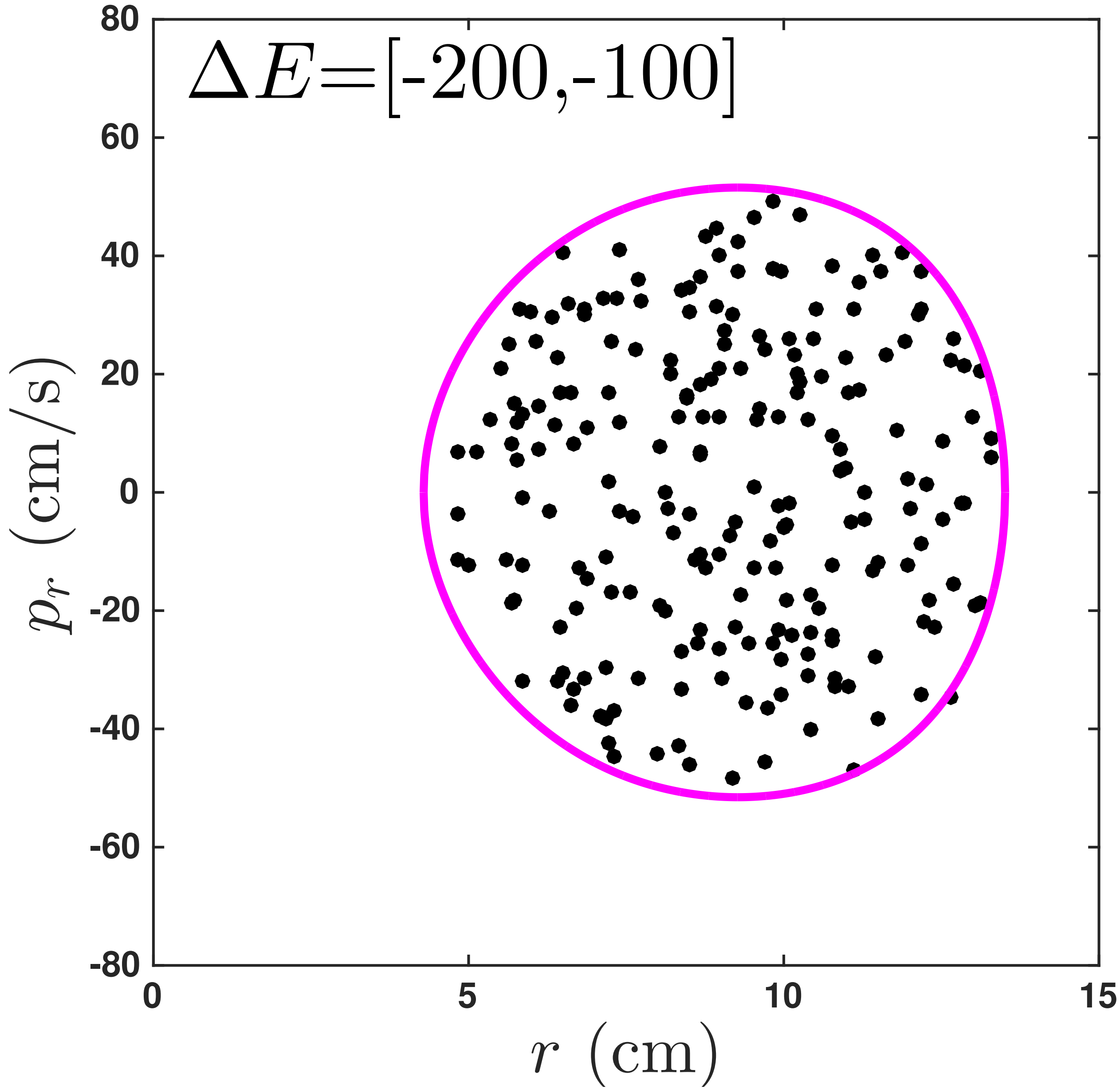

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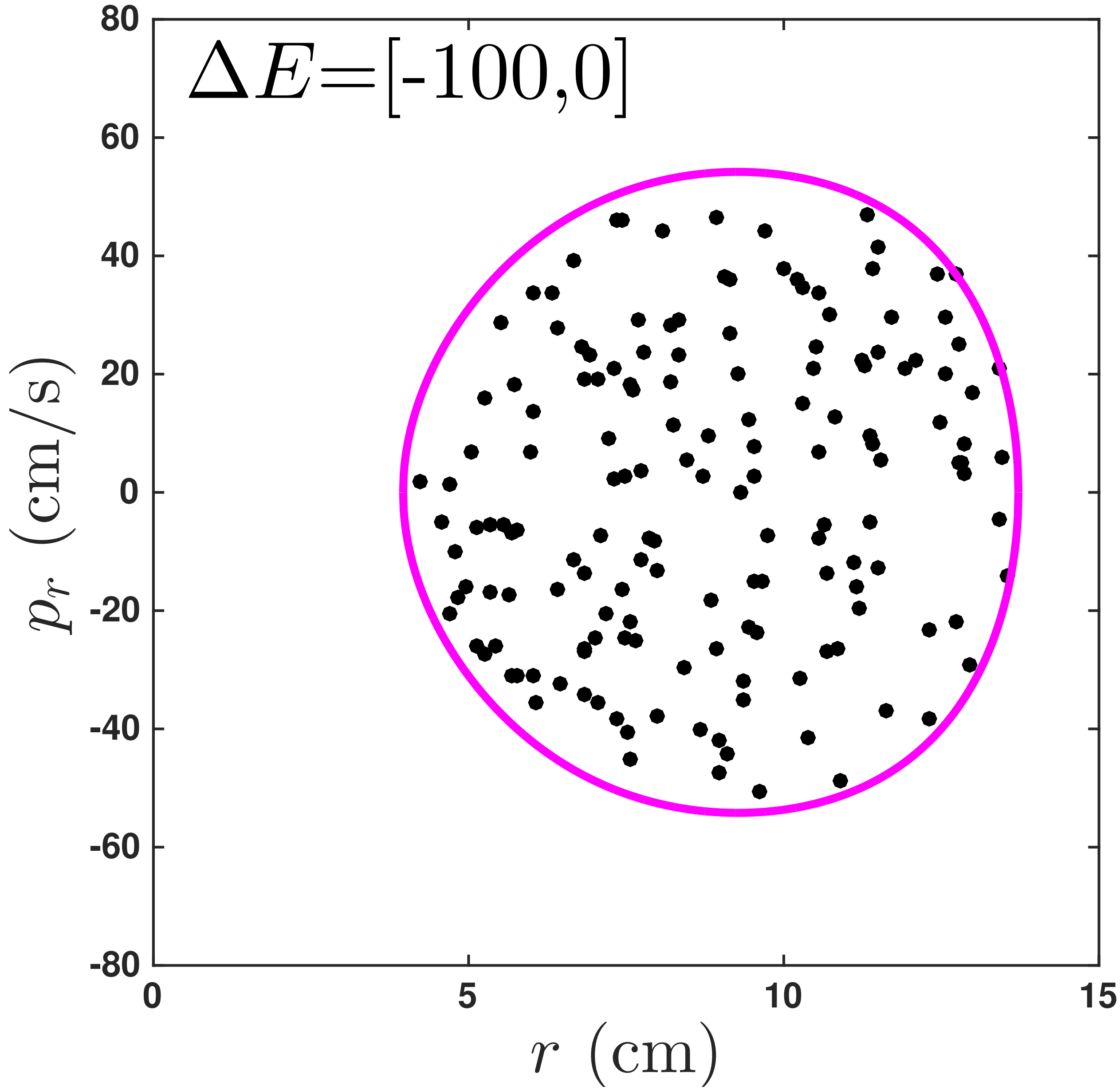

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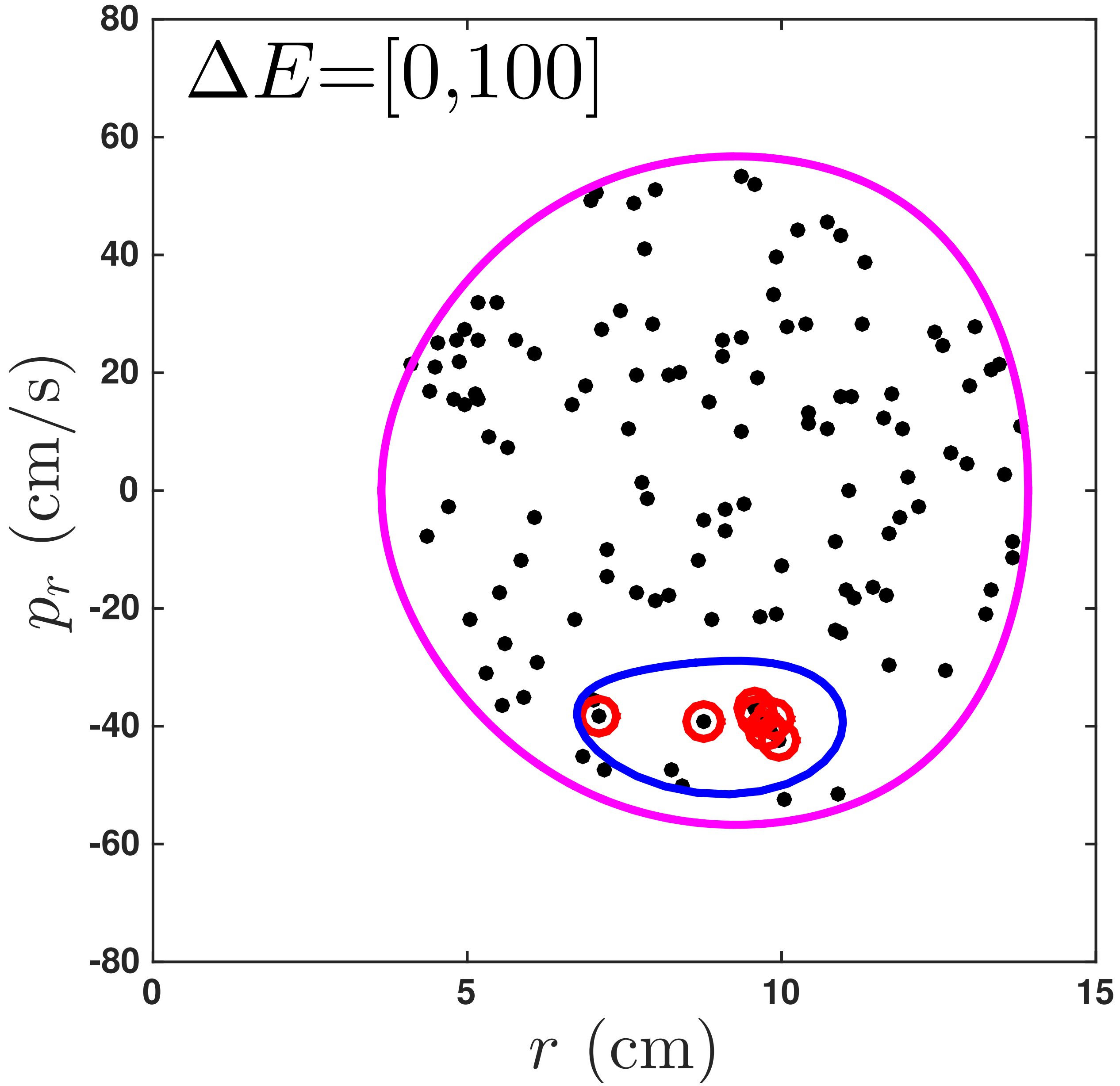

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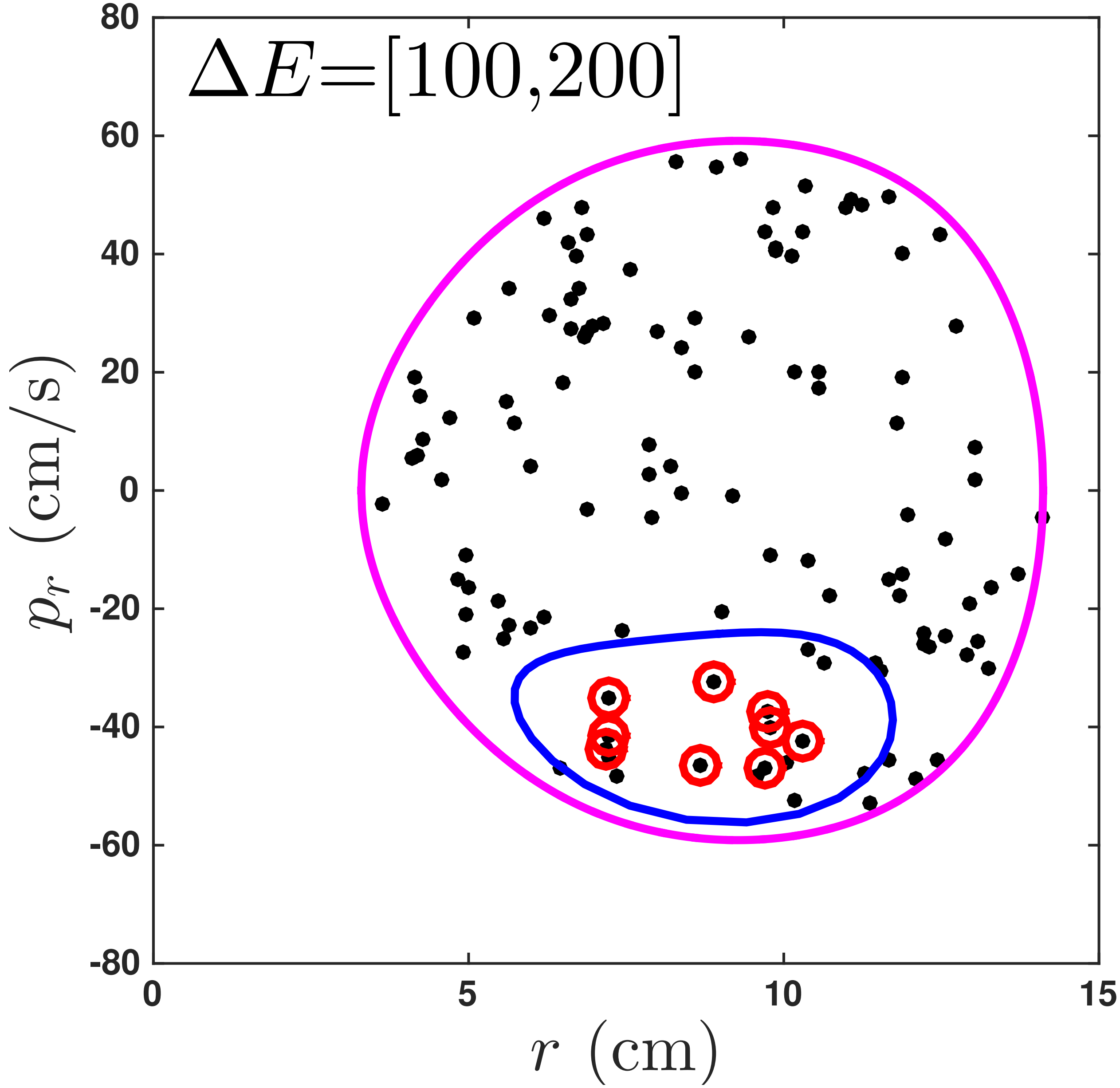

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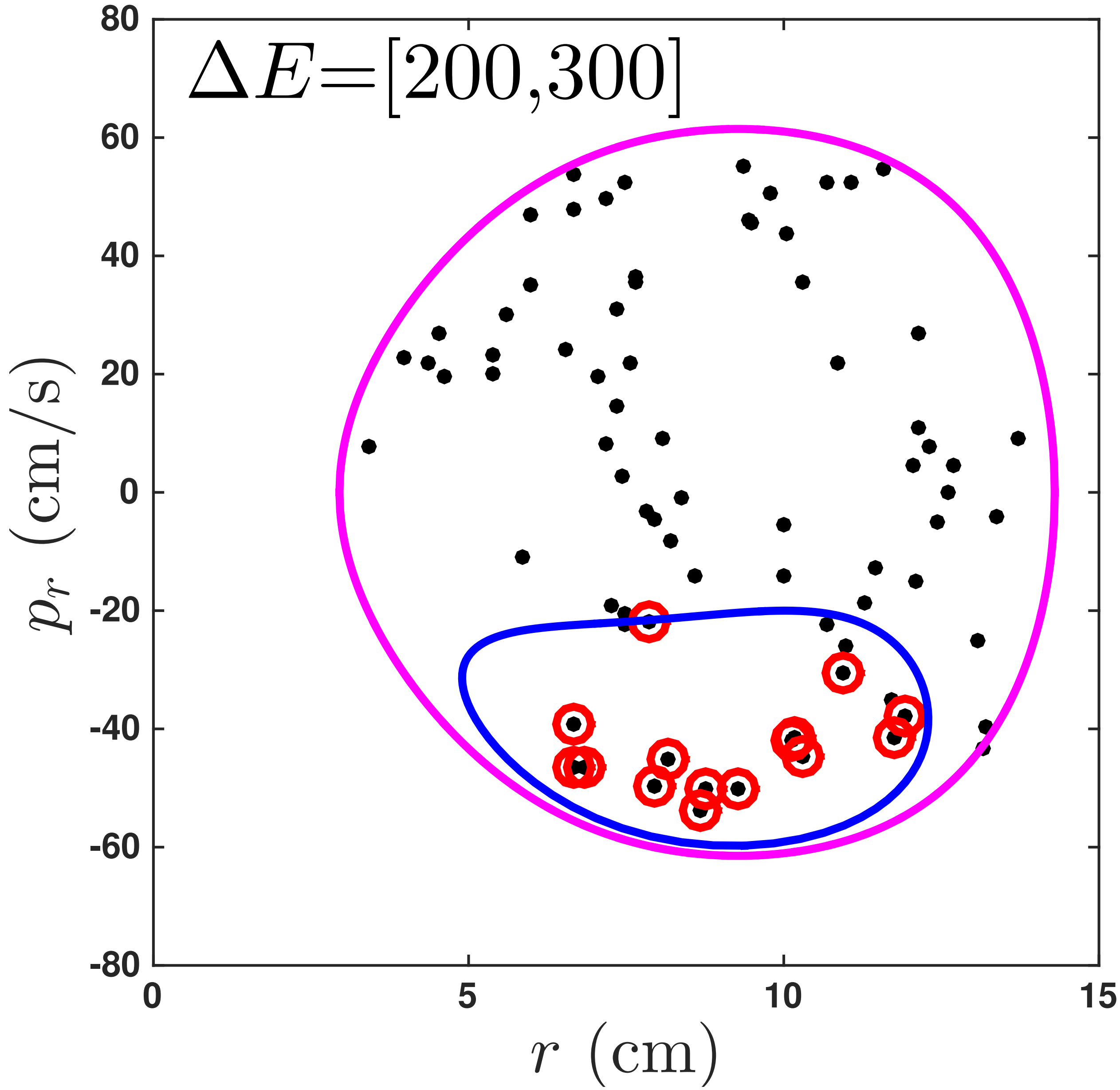

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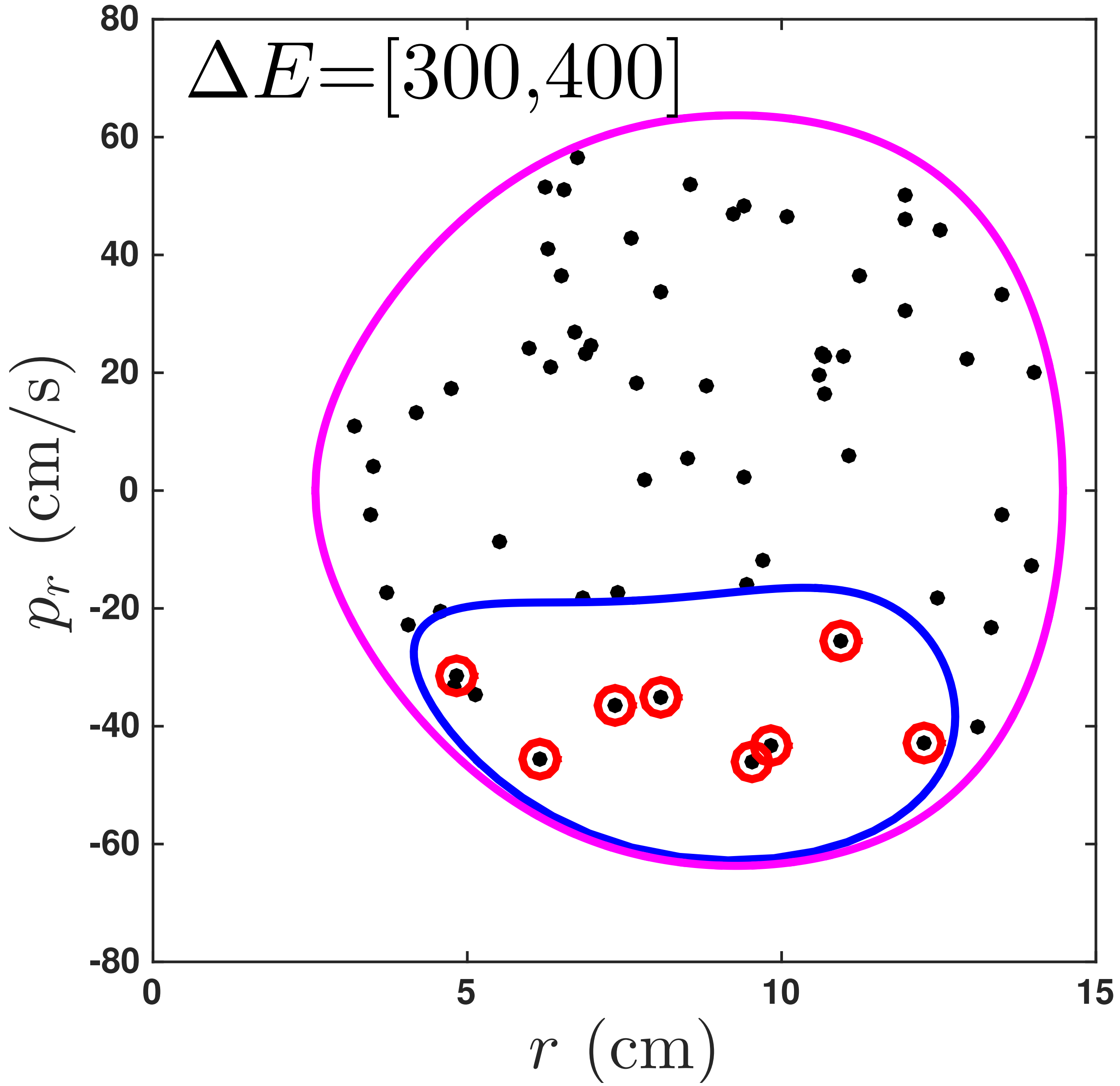

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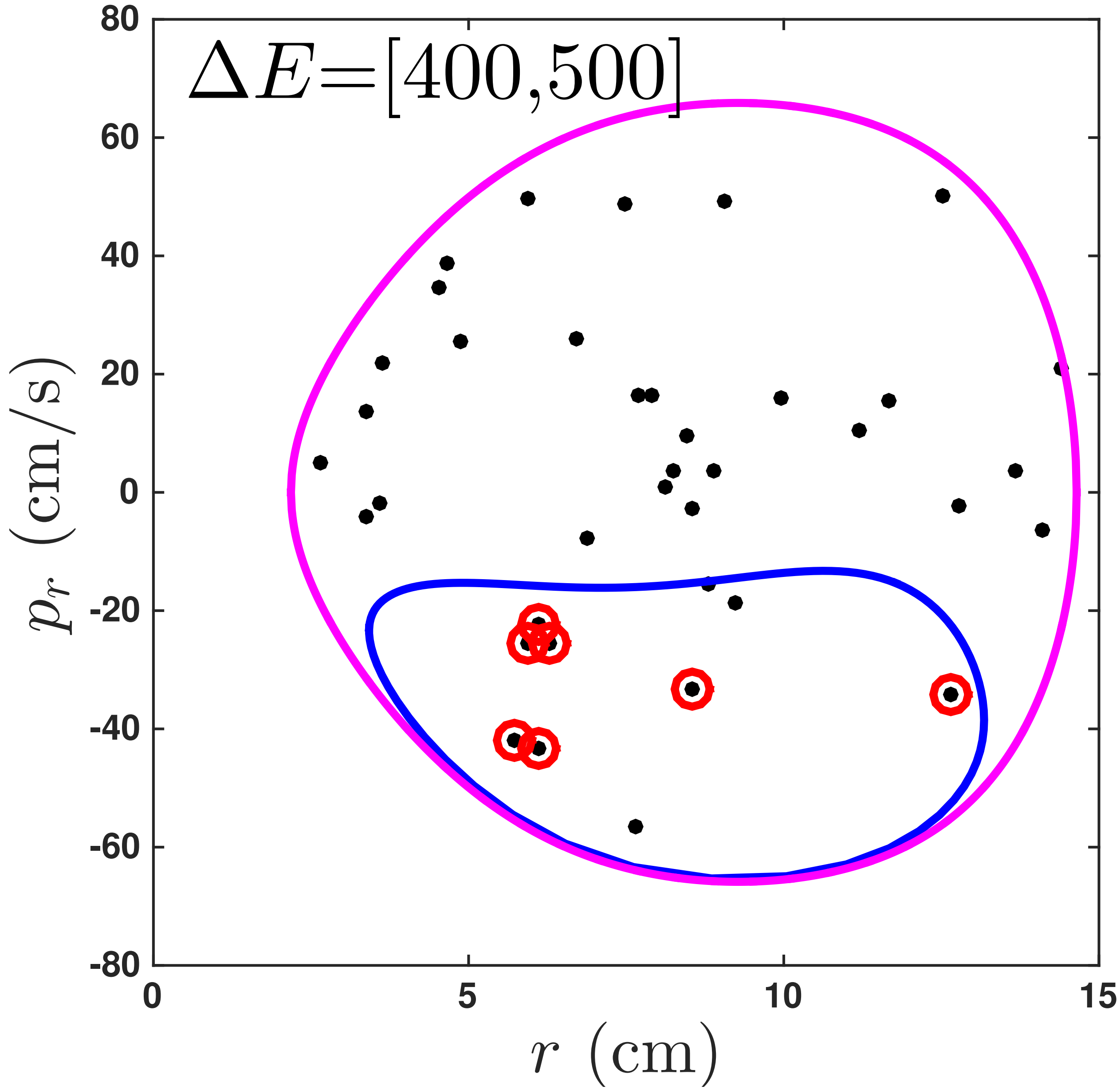

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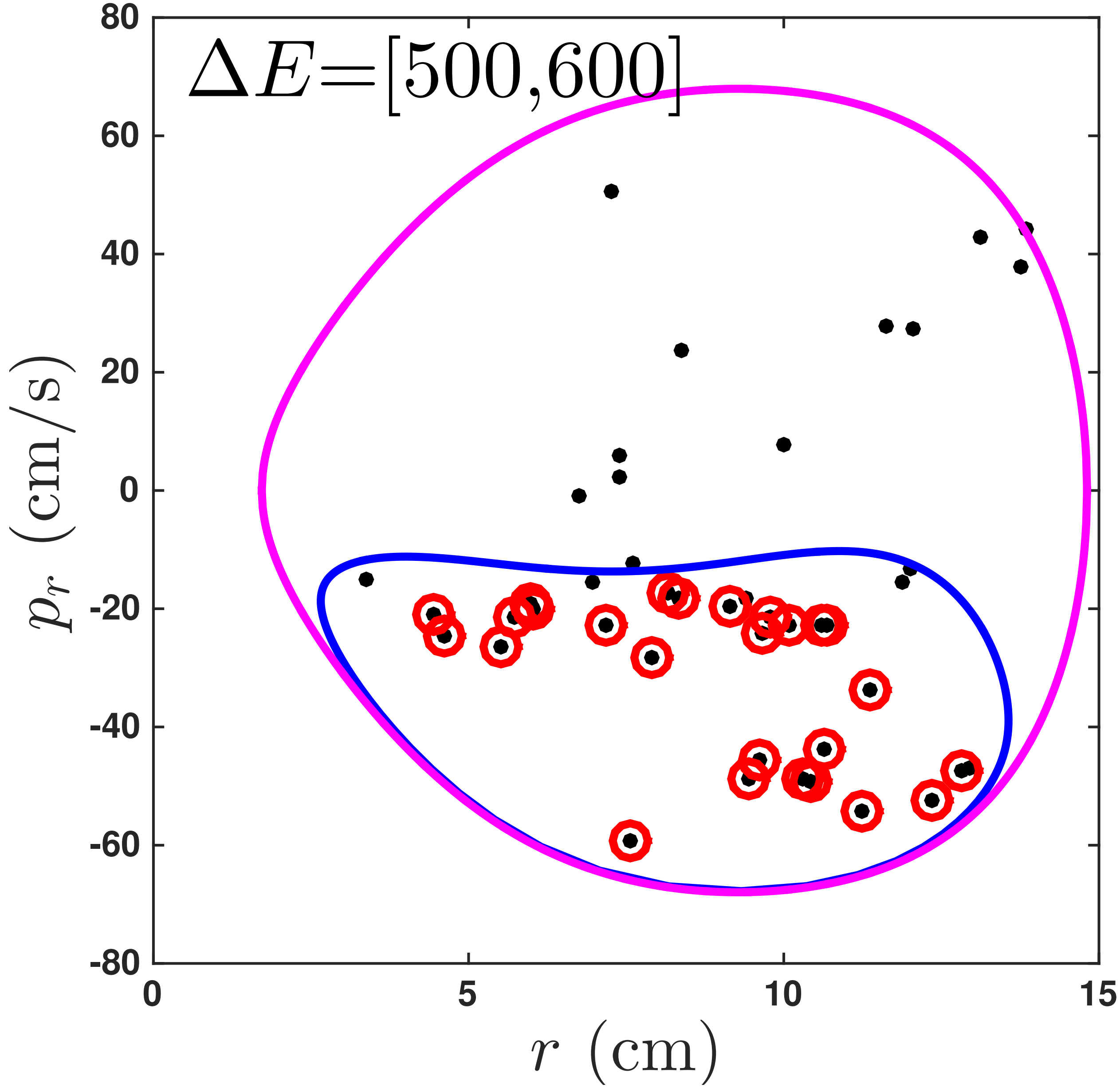

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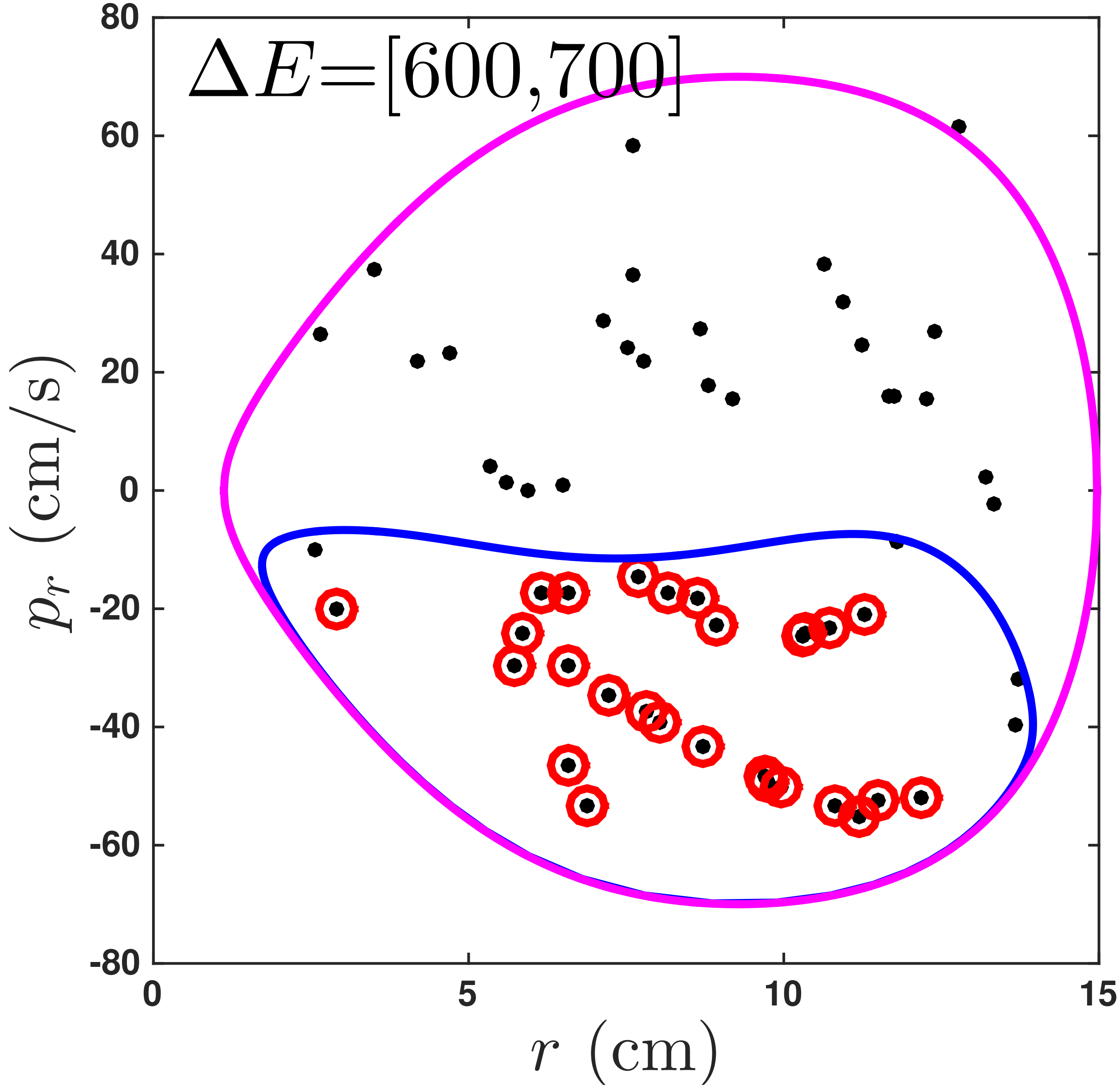

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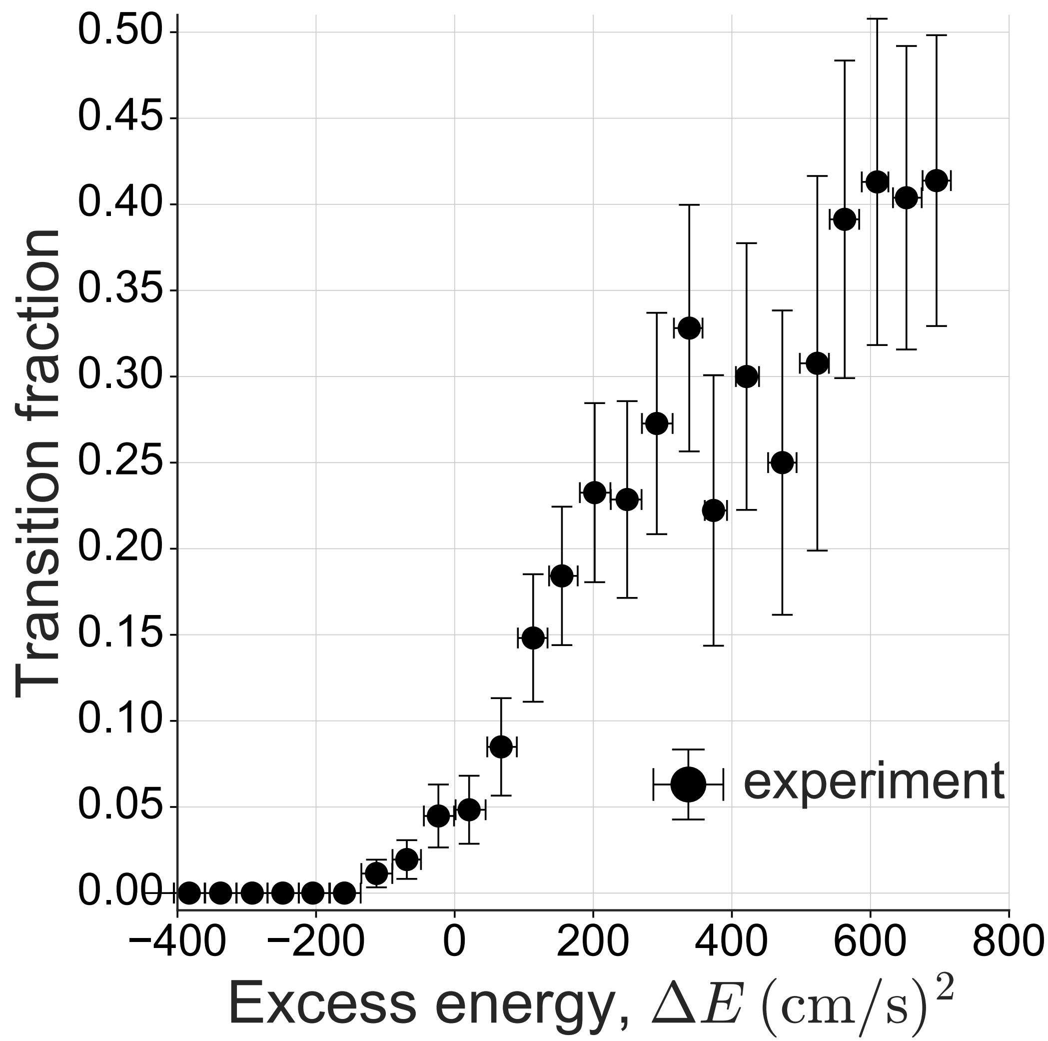

Theory for small excess energy, ΔE

- Area of the transitioning region, the tube cross-section (MacKay [1990])

$$A_{\rm trans} = T_{\rm po} \Delta E$$

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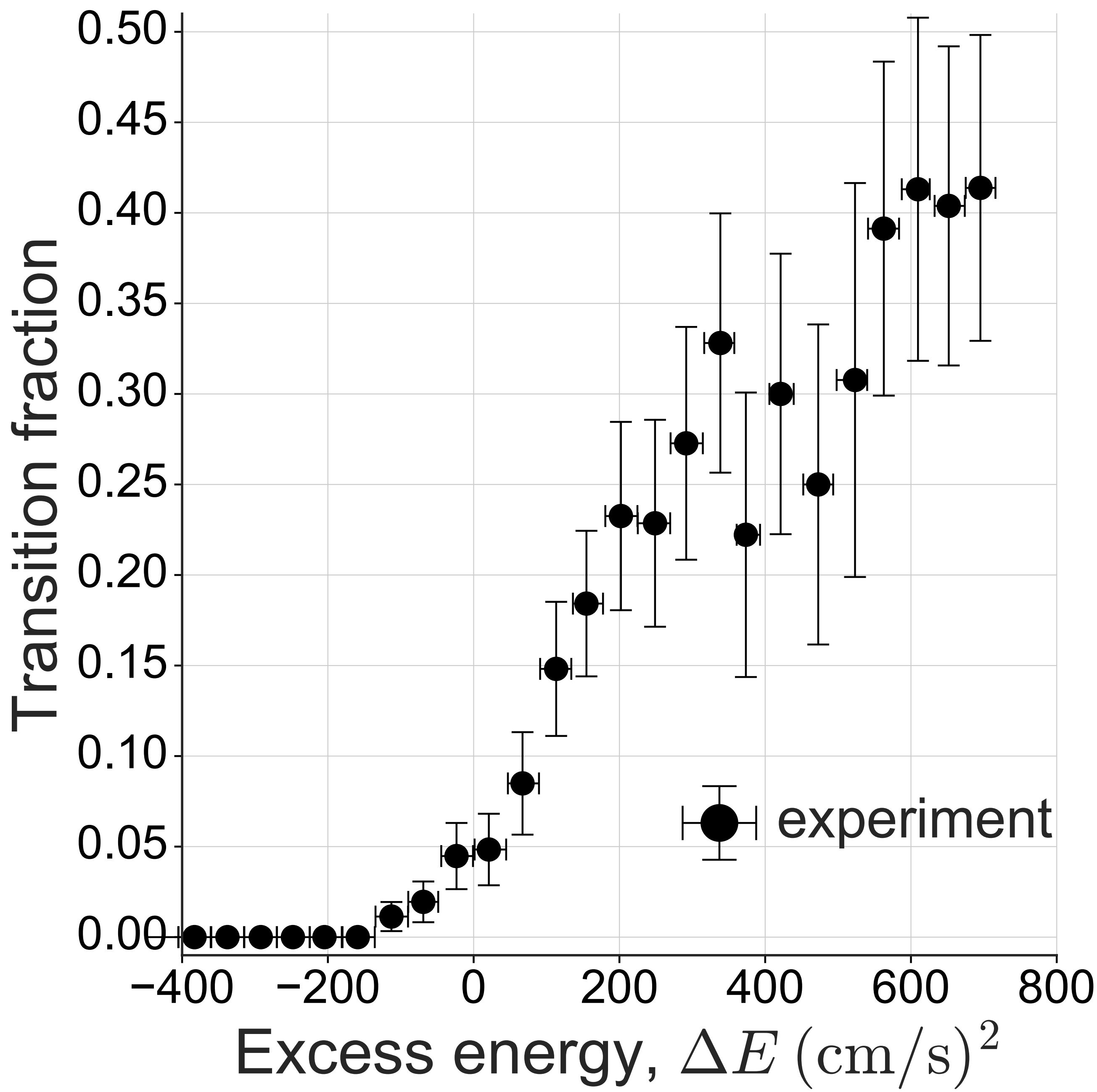

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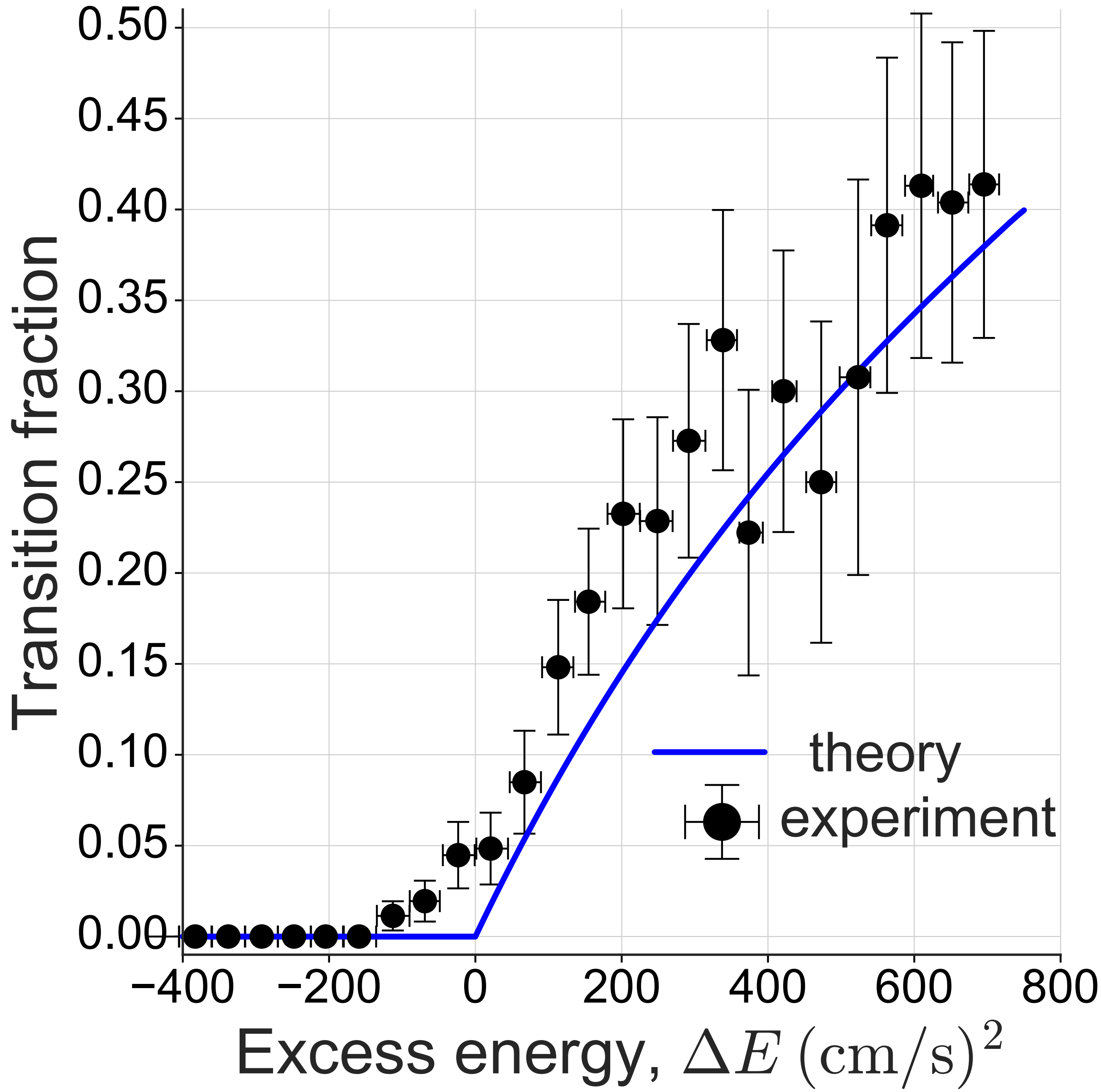

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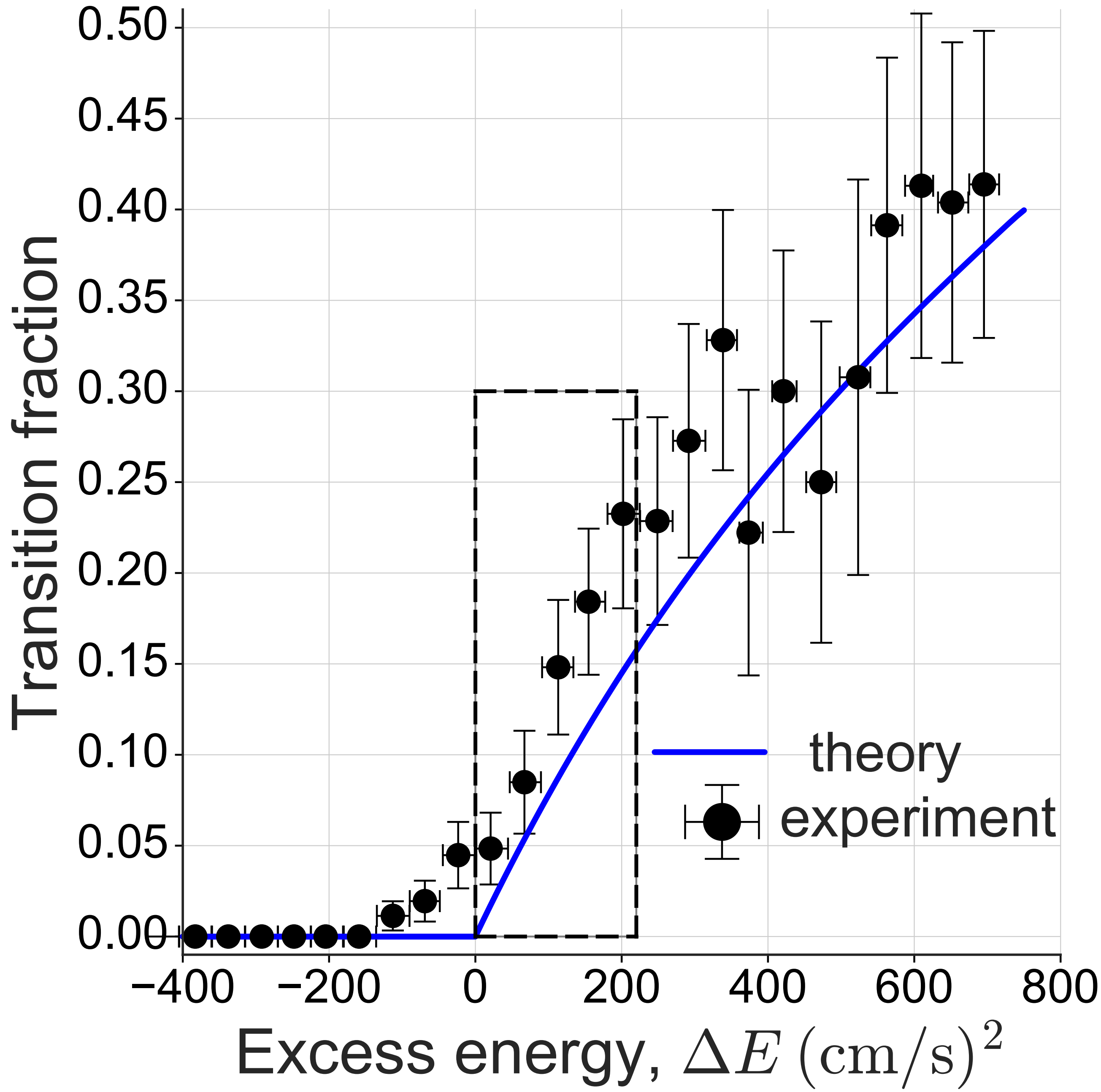

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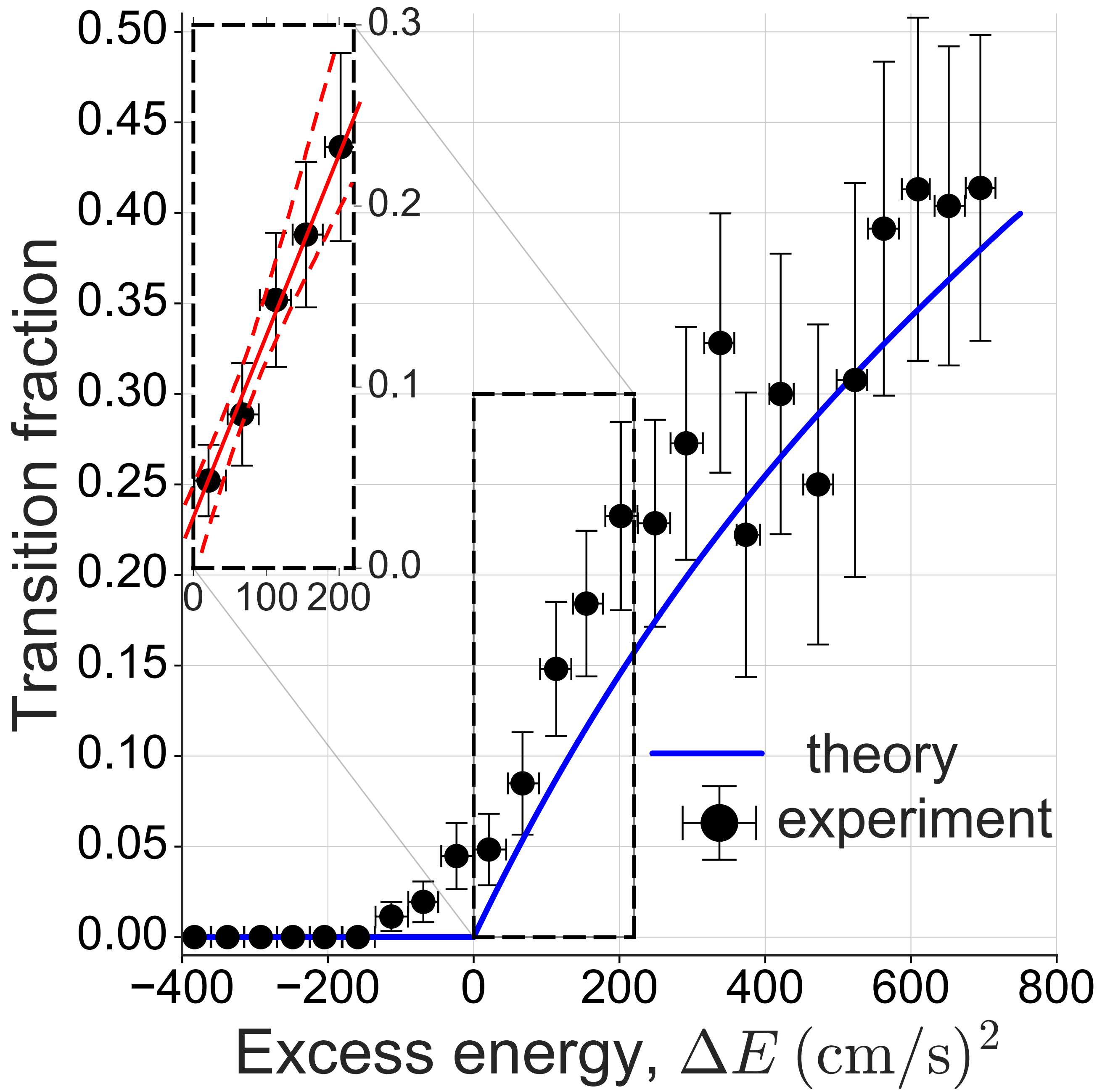

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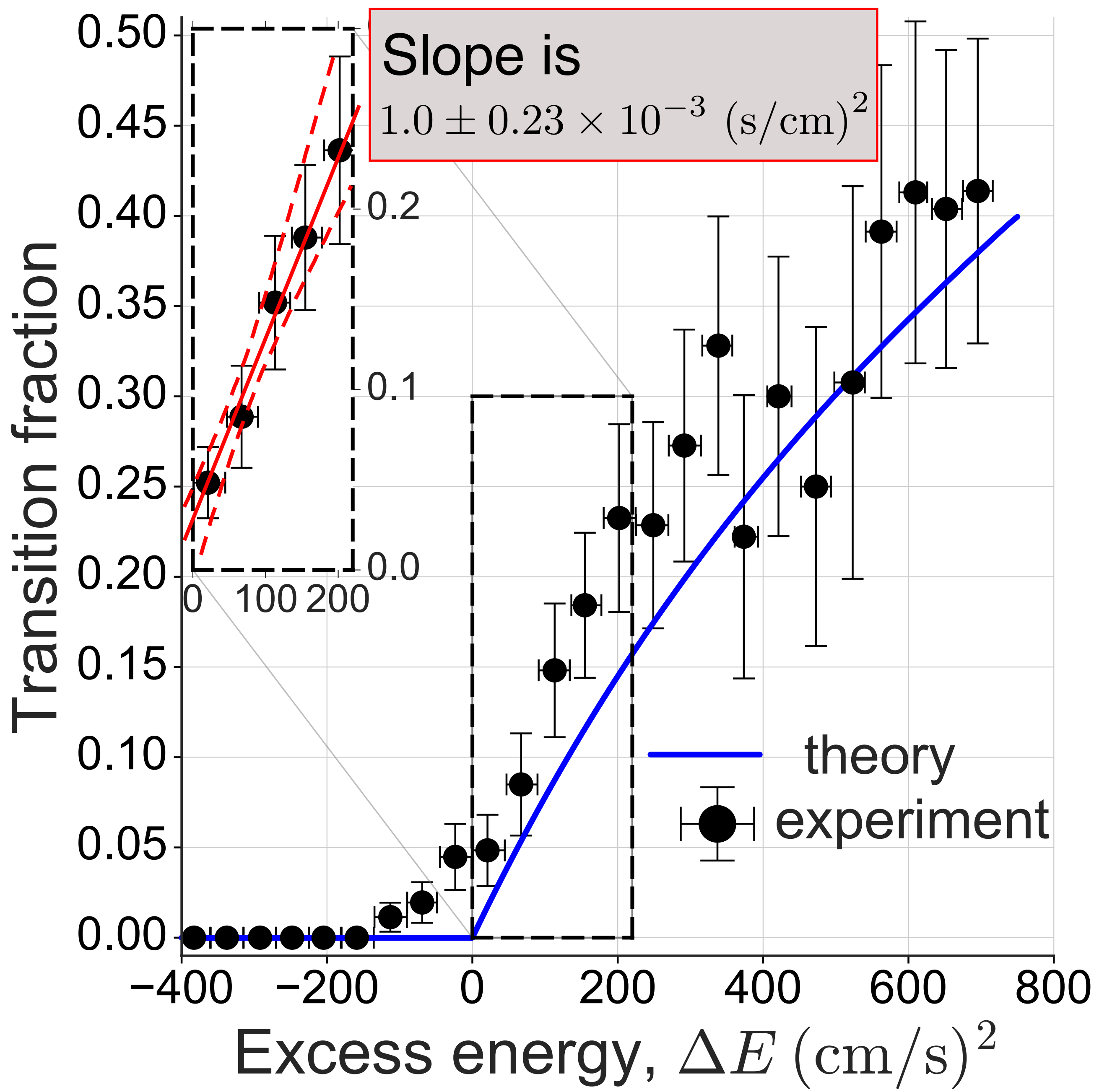
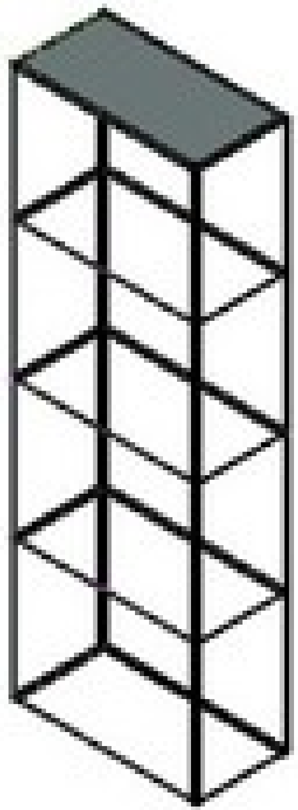
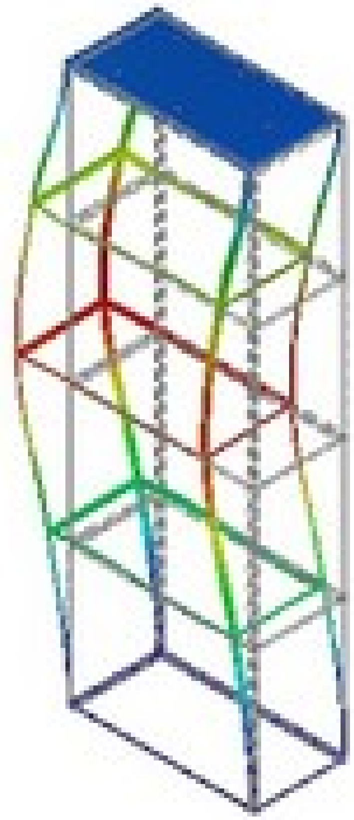
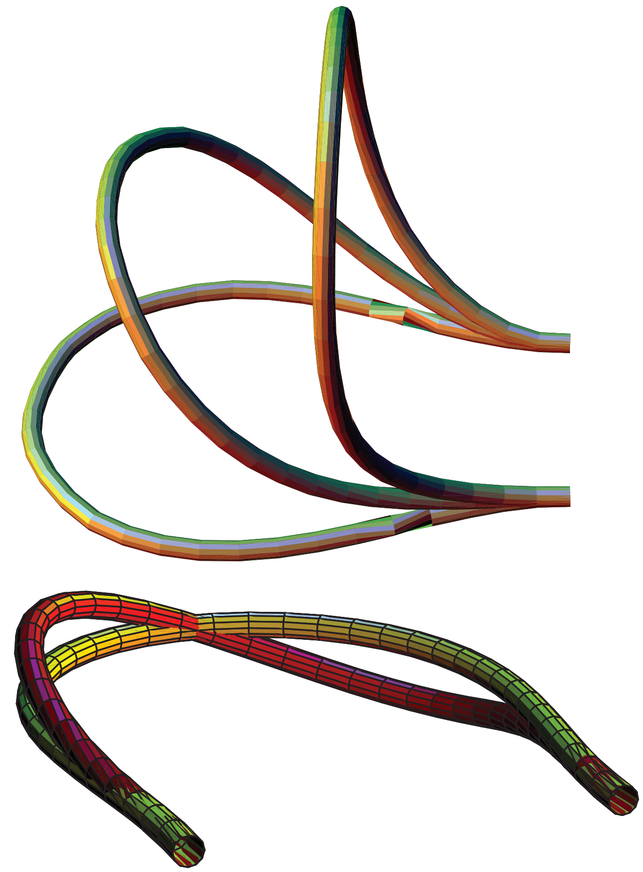


Next steps — structural mechanics

Buckling, bending, twisting, and crumpling of flexible bodies

- adaptive structures that can bend, fold, and twist to provide advanced engineering opportunities for deployable structures, mechanical sensors

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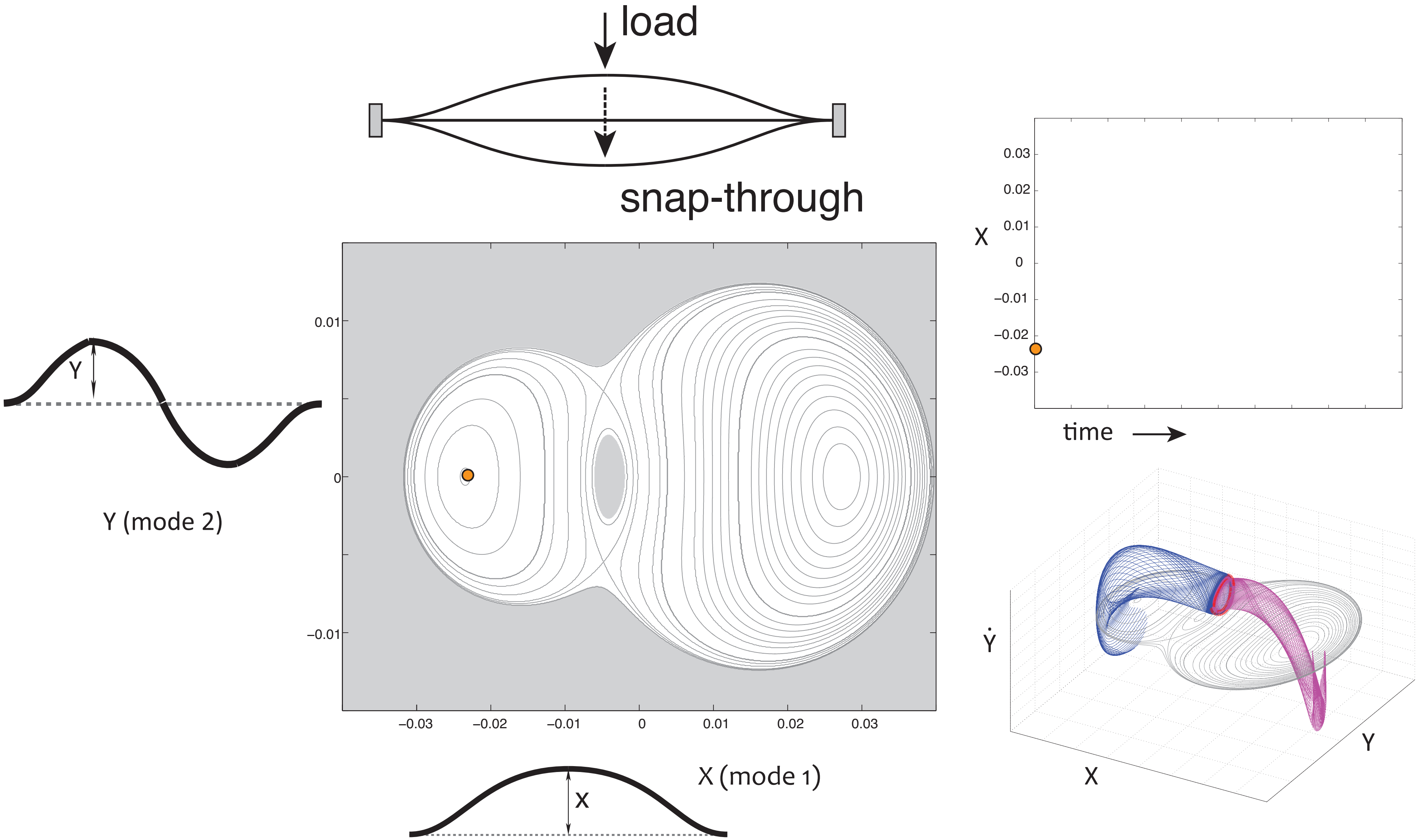

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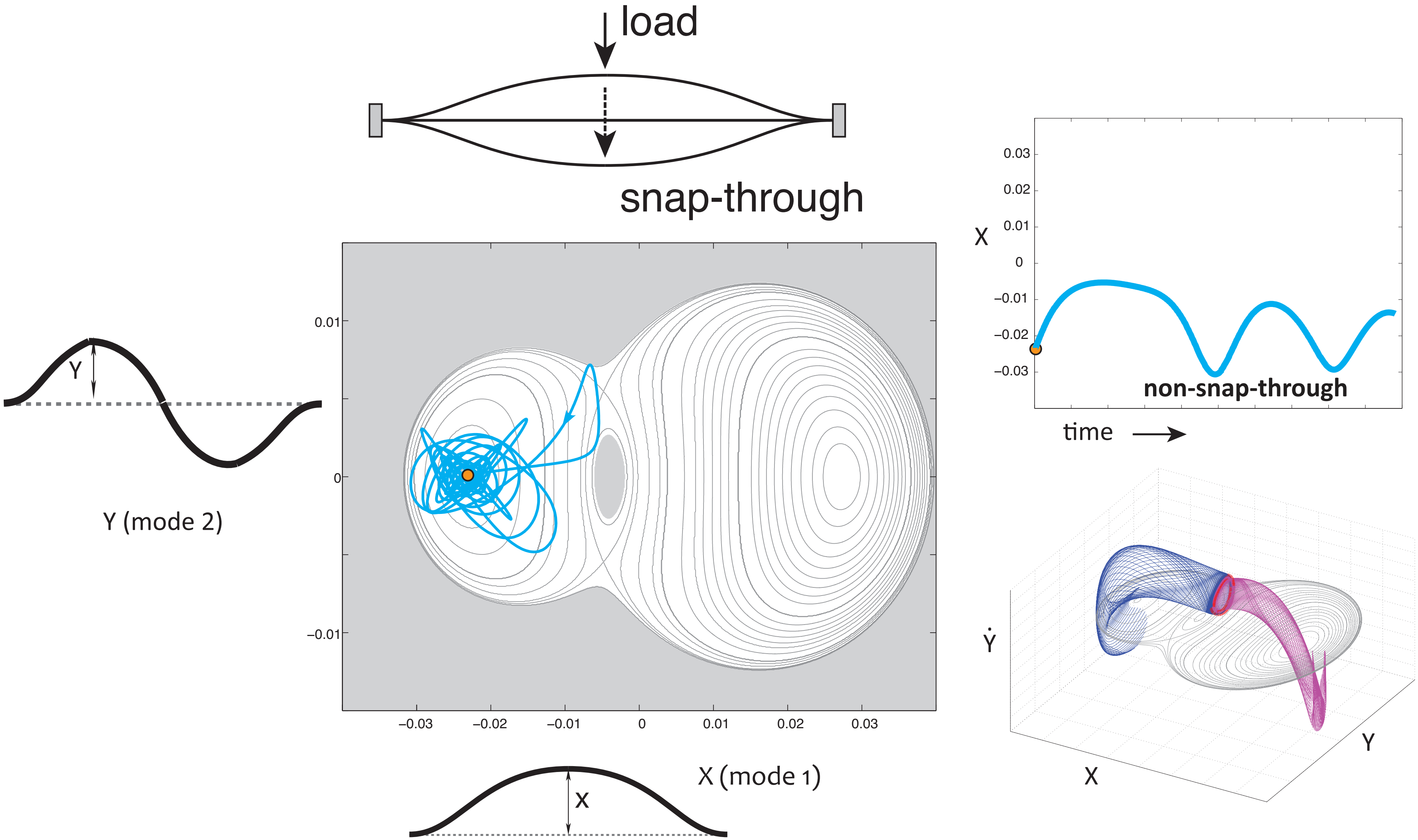

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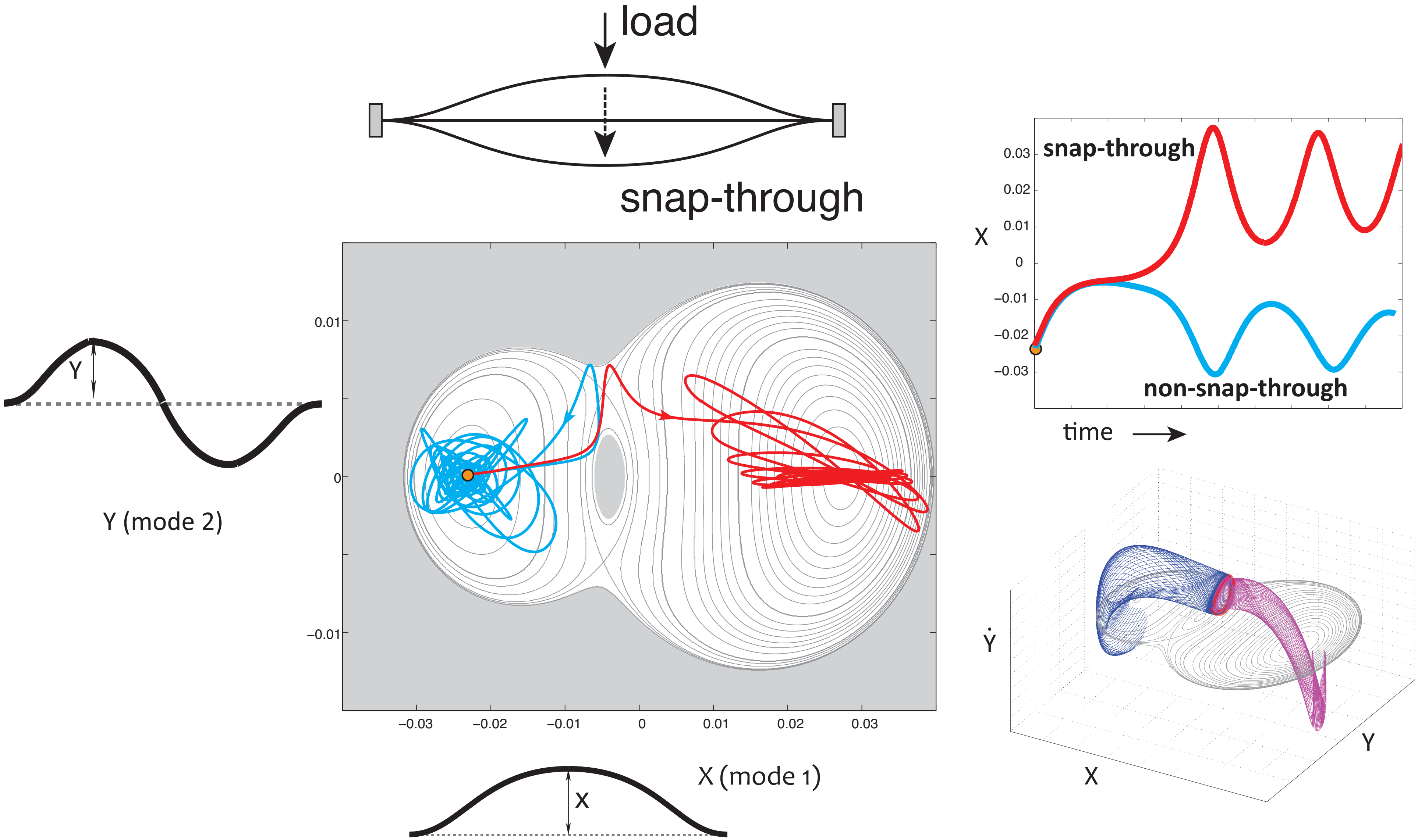

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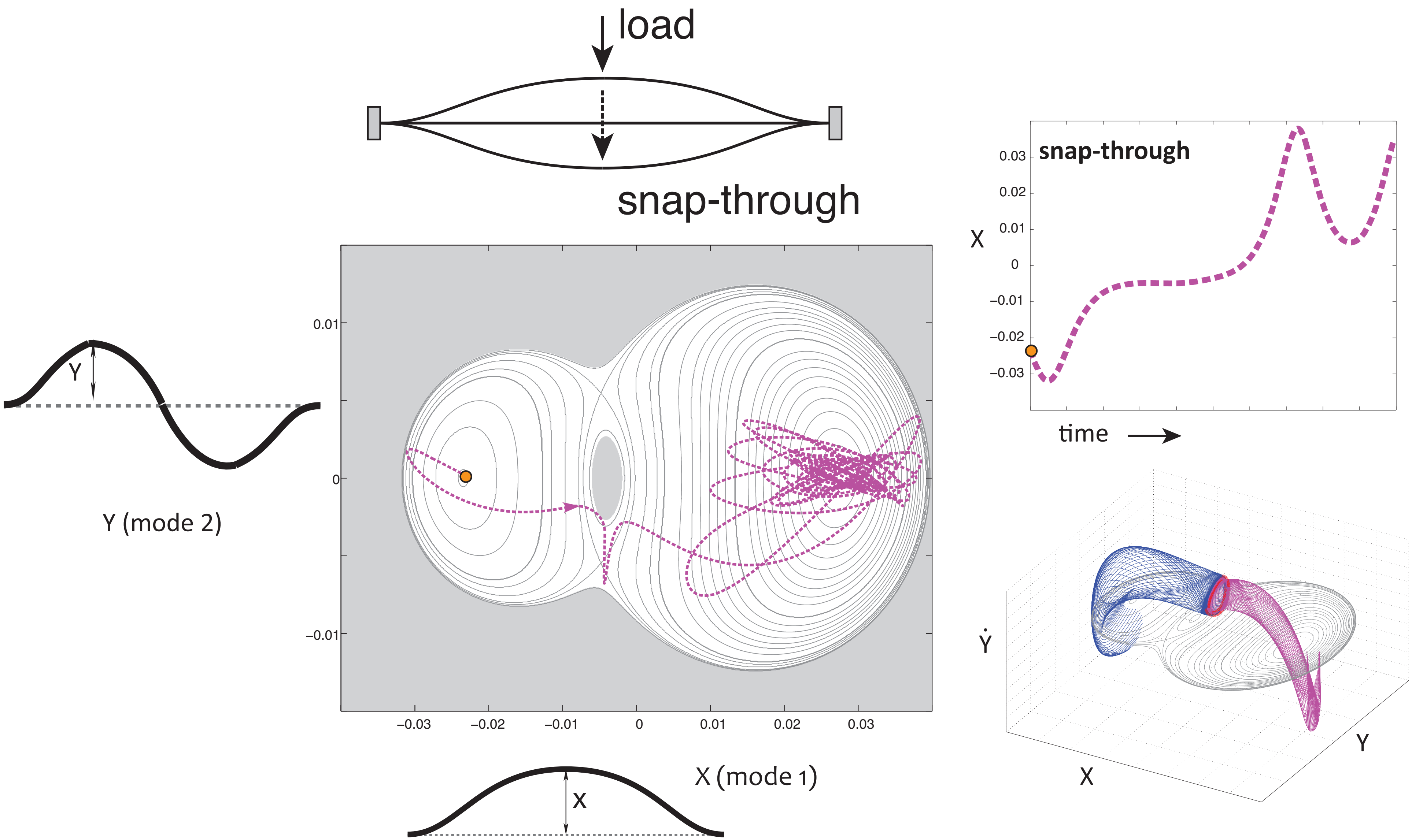

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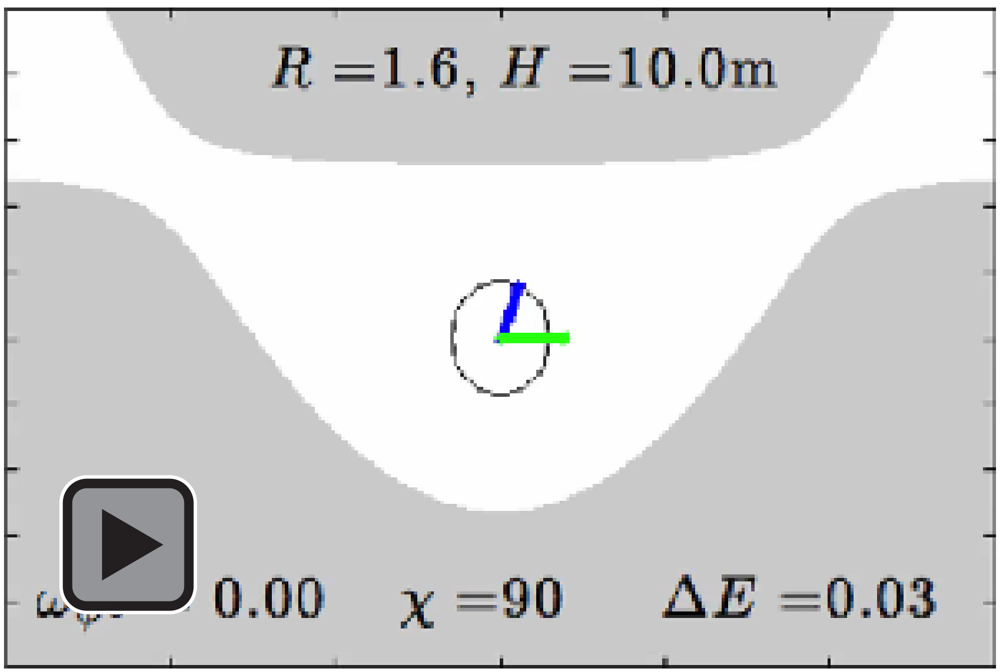

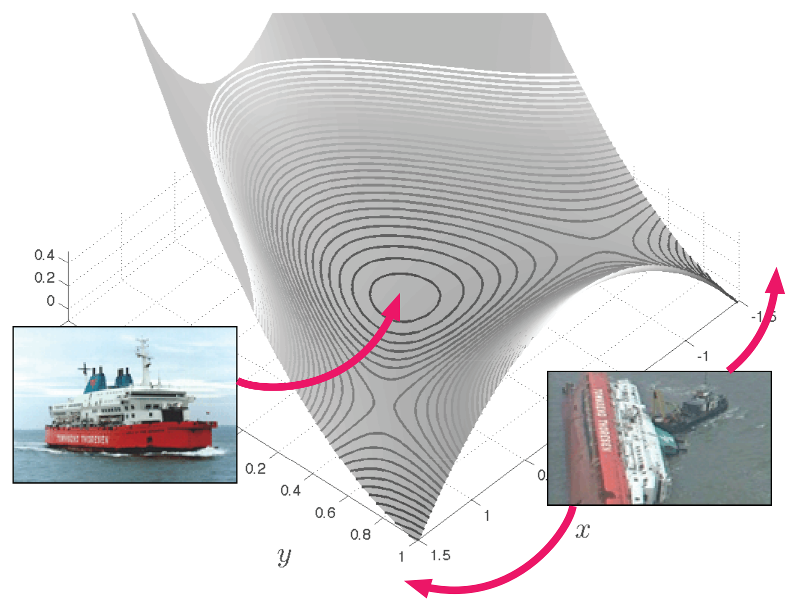

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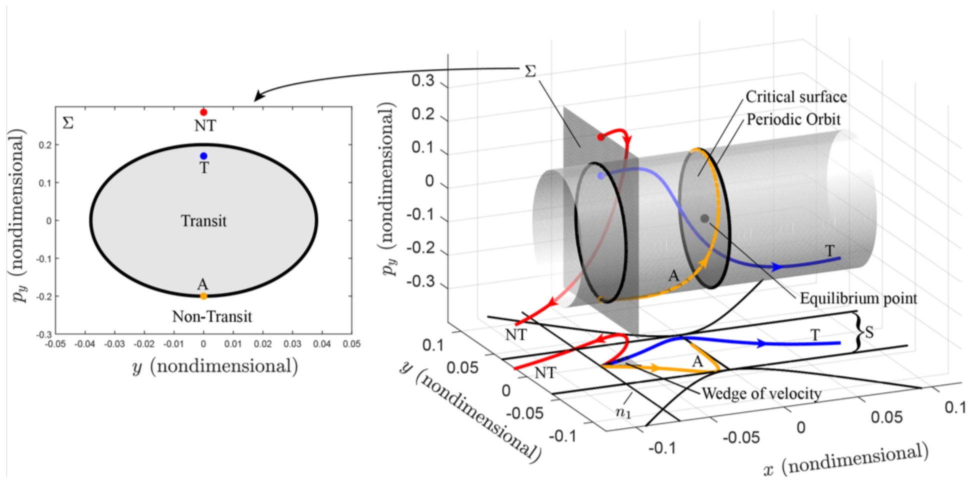

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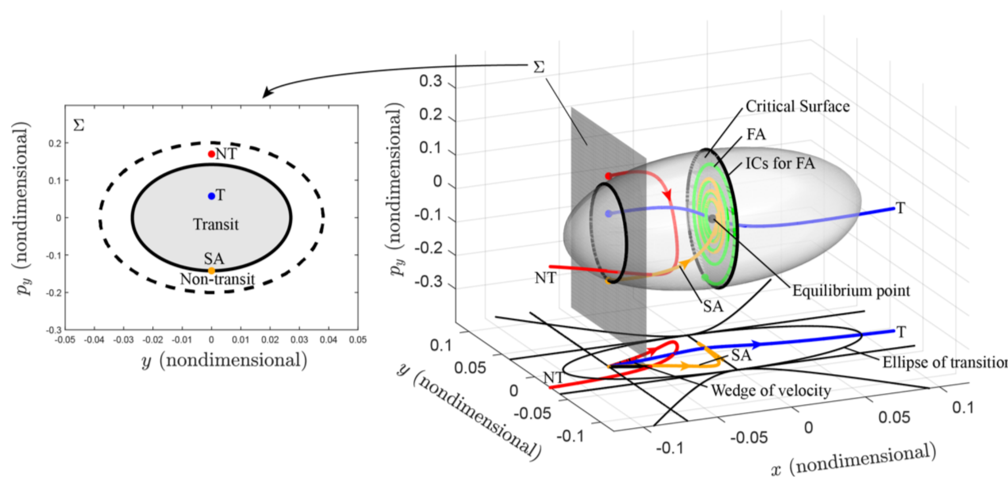

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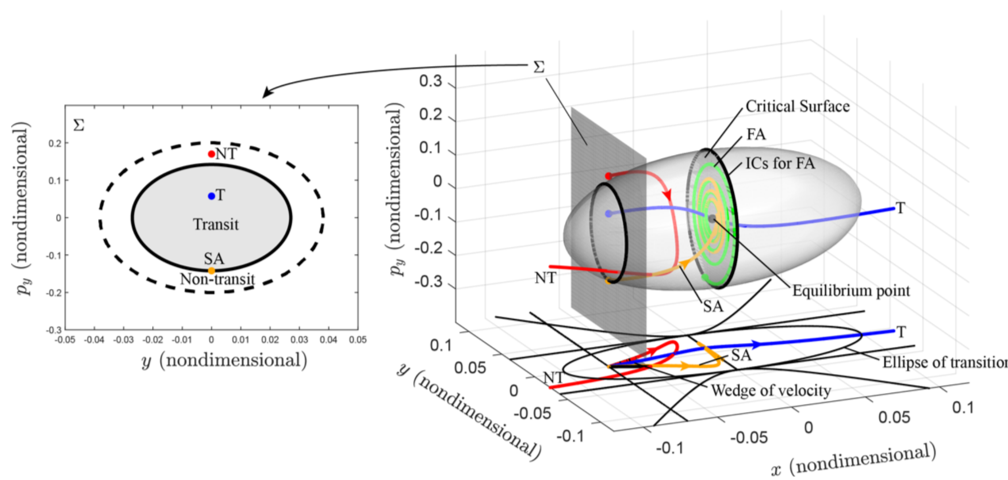

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