Intersections of phase volumes bounded by invariant manifolds

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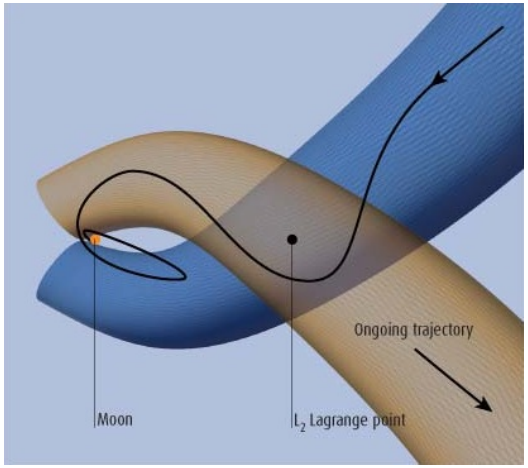
Introduction

- Invariant manifolds of unstable bound orbits act as **separatrices** (codimension 1 surfaces)

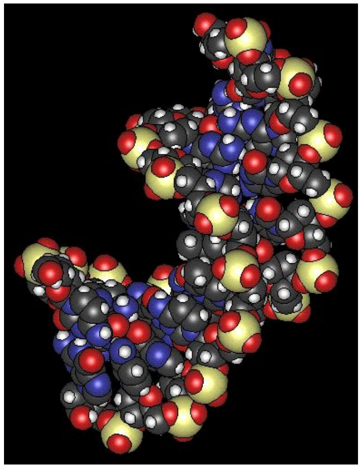
- Determine **transport**, e.g., collisions, transitions

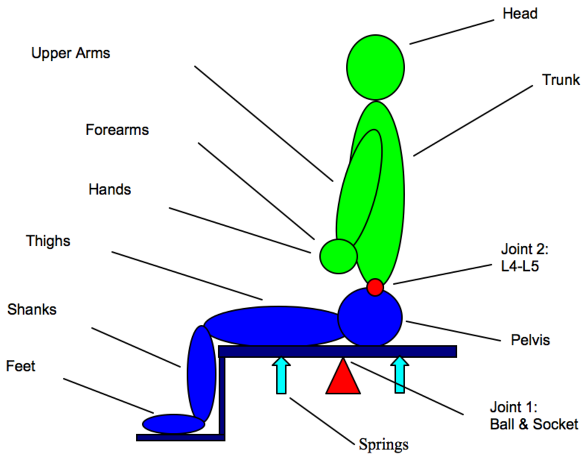
- Use **analytical map** of phase space where appropriate



Introduction

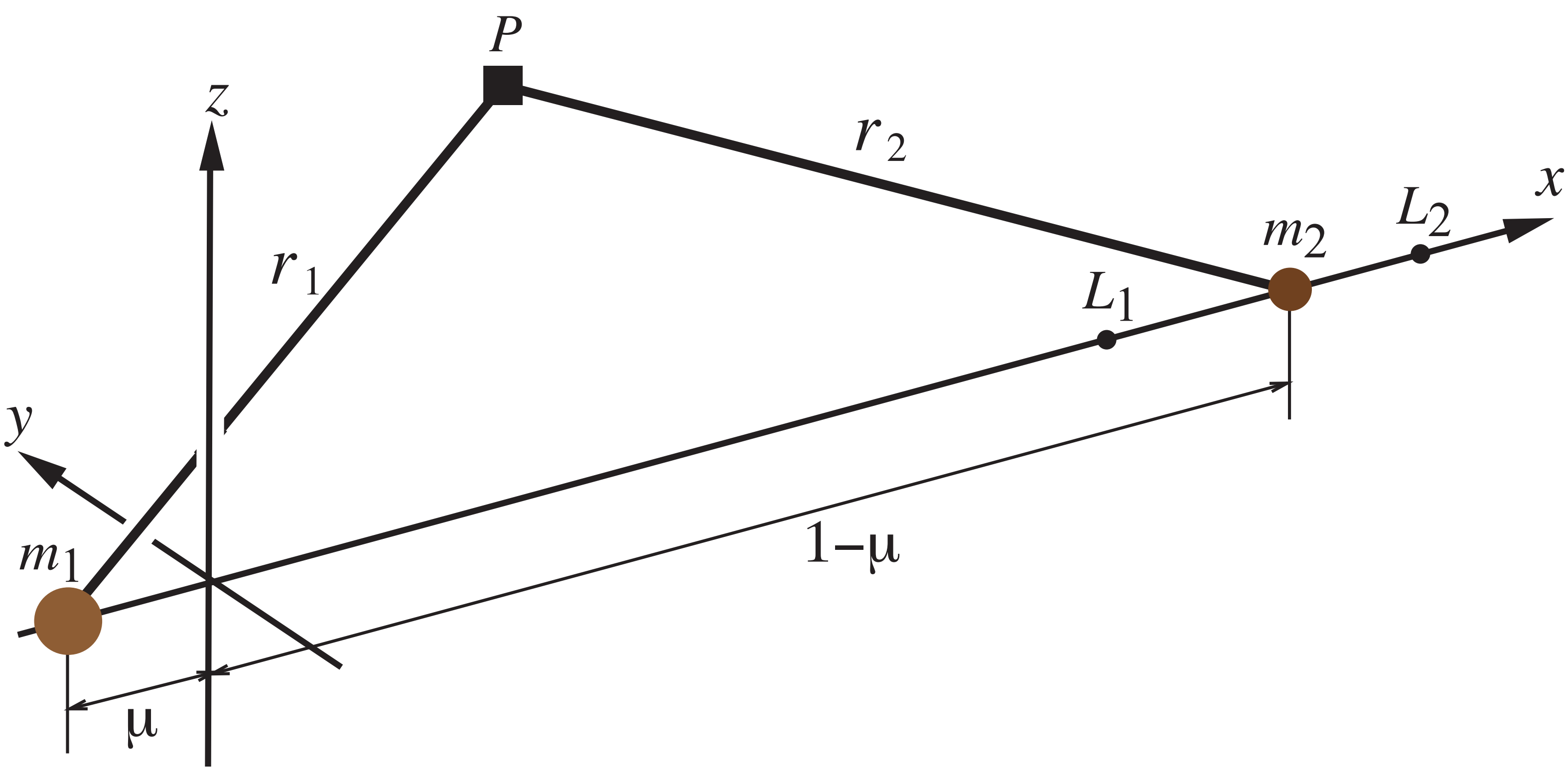
- **Value-added:** results apply to similar problems in e.g., chemistry, biomechanics, boat capsize





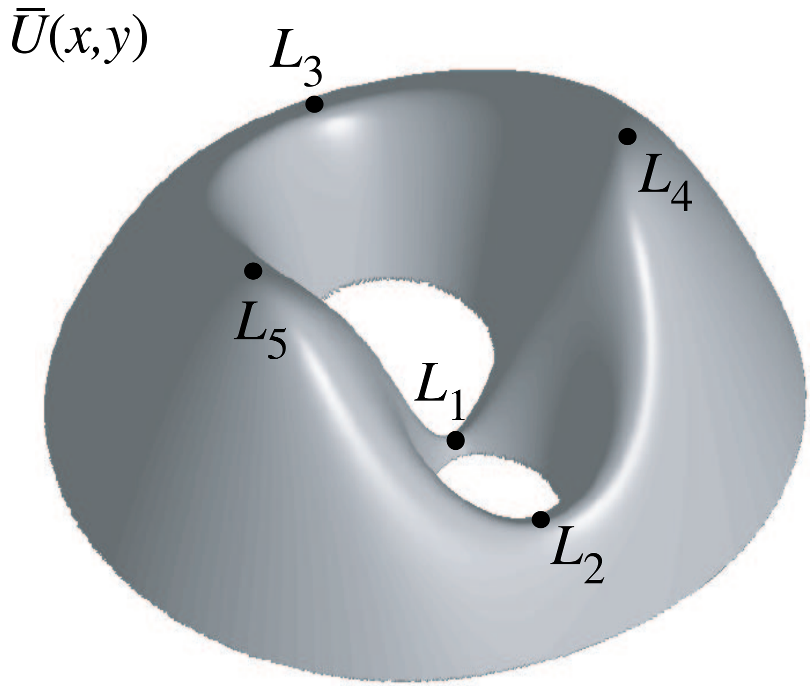
3-Body Problem

- ***Circular restricted 3-body problem***
  - Two important landmarks, the unstable points L_1, L_2



3-Body Problem

- Equations of motion in rotating frame describe P moving in an effective potential plus coriolis force (goes back to work of Jacobi, Hill, etc)



Effective Potential

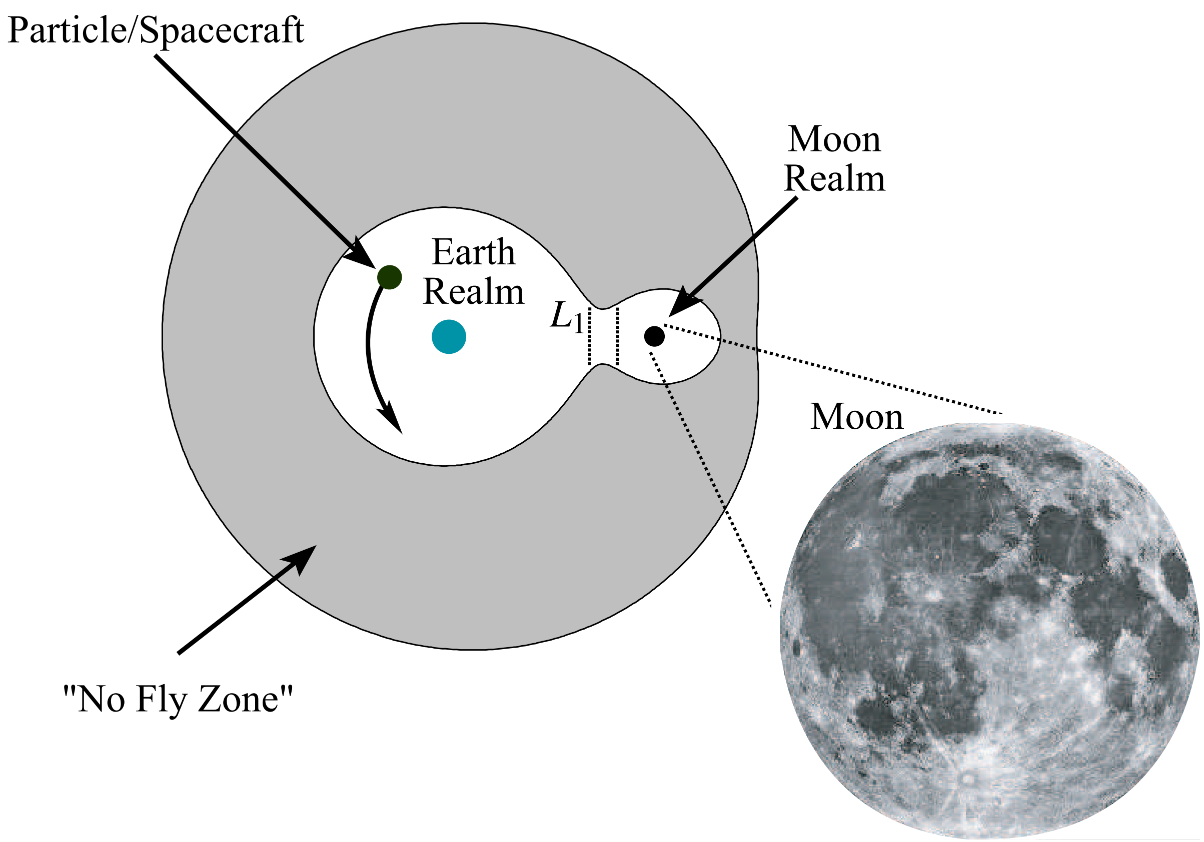
Motion in energy surface

- Hamiltonian function $H(q,p)$
- **Energy surface** of energy E is codim-1

$$\mathcal{M}(E) = \{(q,p) \mid H(q,p) = E\}$$

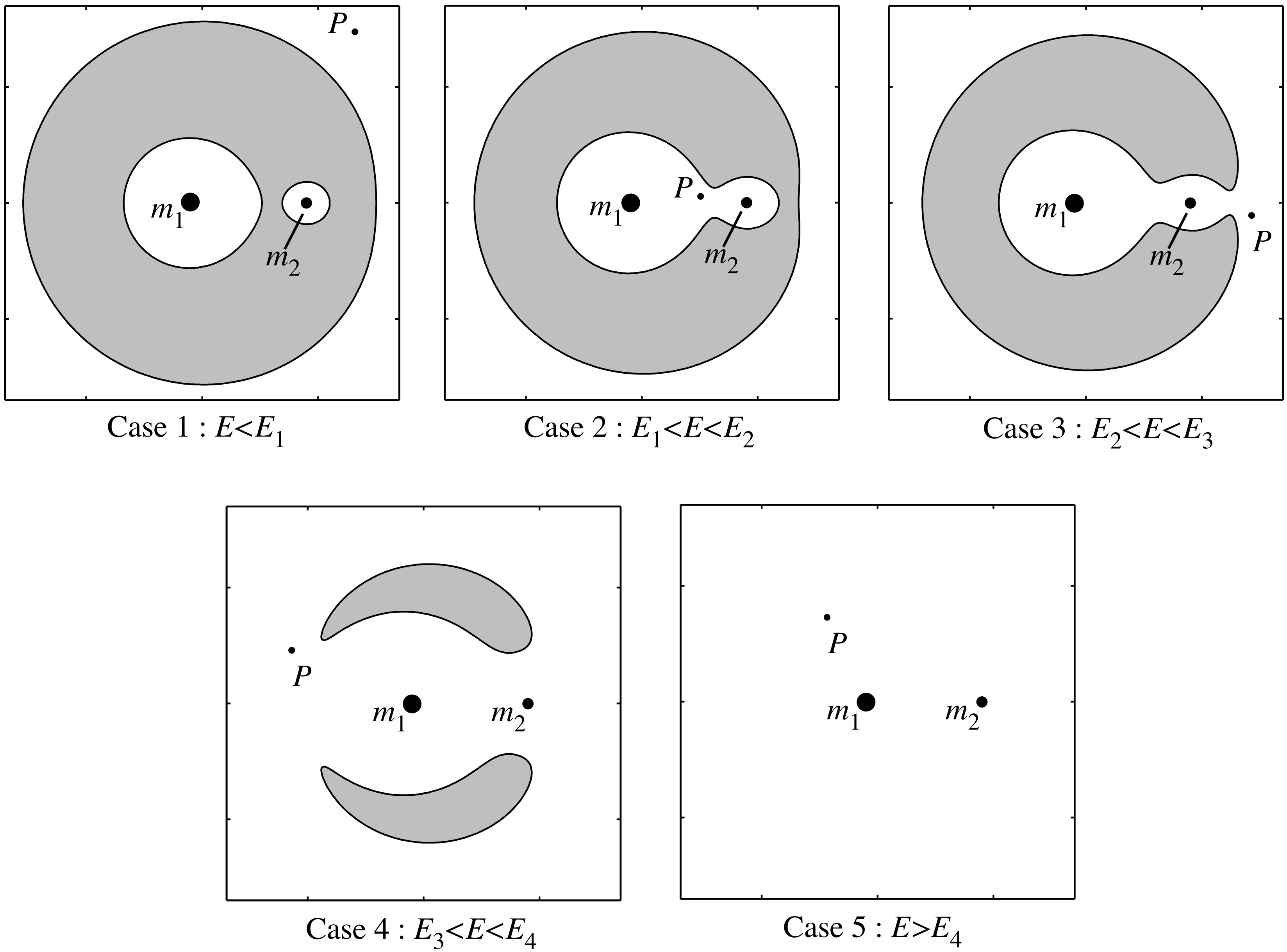
- In 2 d.o.f., 3D surfaces foliating the 4D phase space (in 3 d.o.f., 5D energy surfaces)

Realms of possible motion

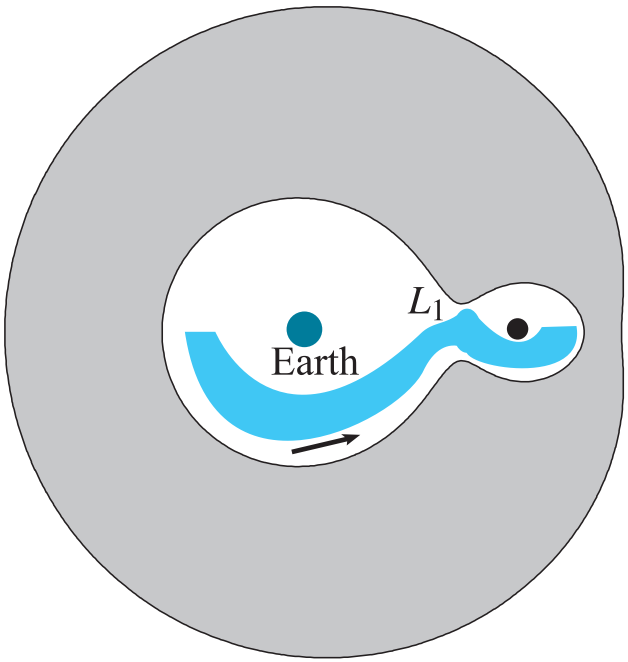


- $\mathcal{M}_\mu(E)$ partitioned into three **realms**
  e.g., Earth realm = phase space around Earth
- Energy E determines their connectivity

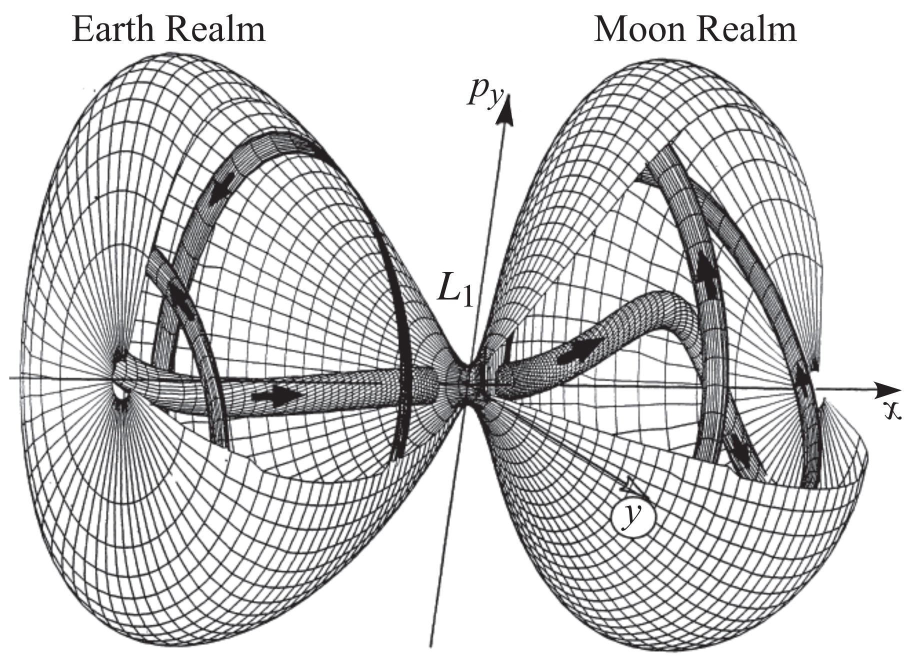
Realms of possible motion



Realms and tubes



Position Space

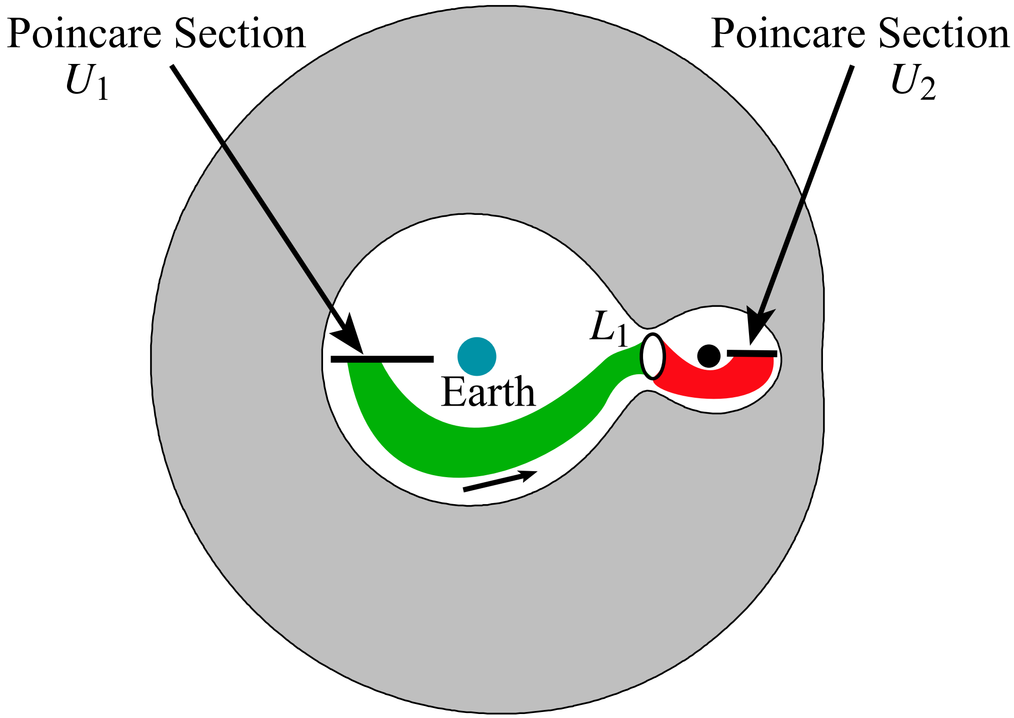


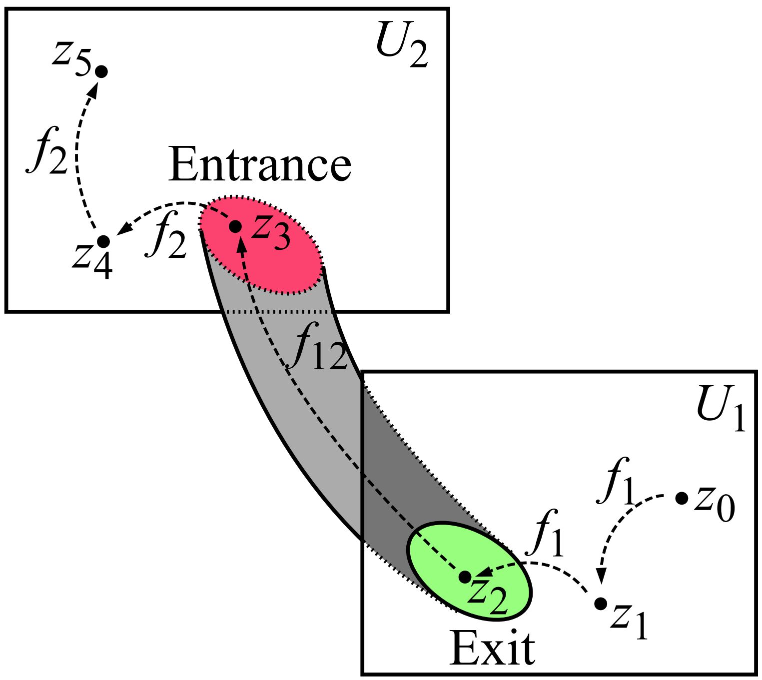
Phase Space (Position + Velocity)

- Realms connected by **tubes** in phase space $\simeq S^k \times \mathbb{R}$
  — Conley & McGehee, 1960s, found these locally for planar case, speculated on use for **"low energy transfers"**

Multi-scale dynamics

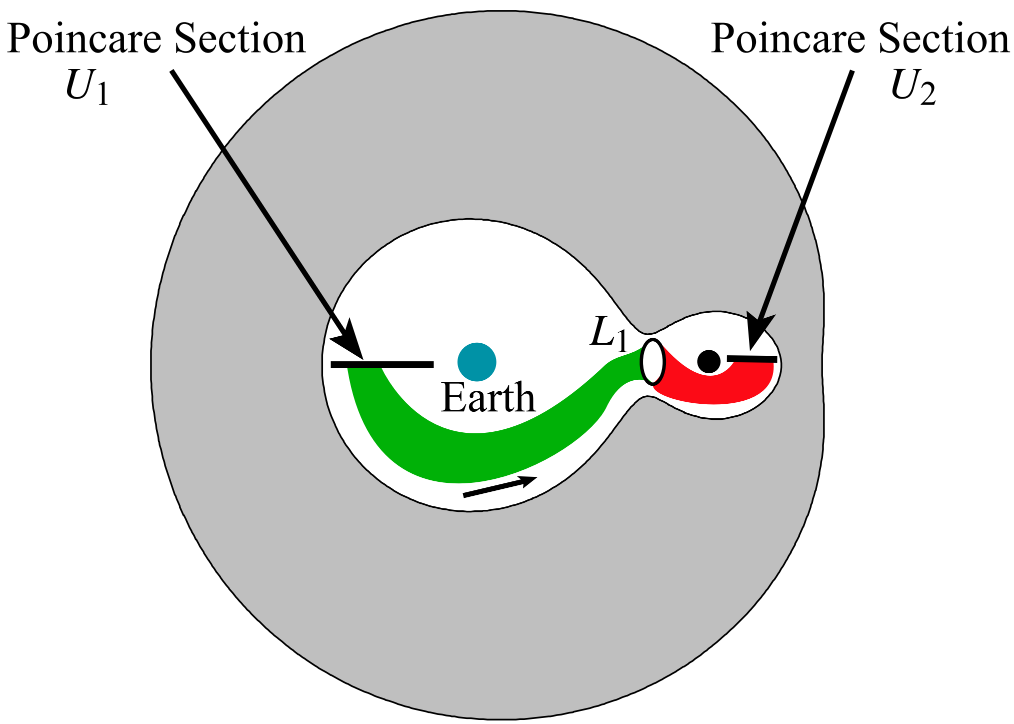
- Slices of energy surface: Poincaré sections U_i
- Tube dynamics: evolution **between** U_i
- What about evolution **on** U_i?

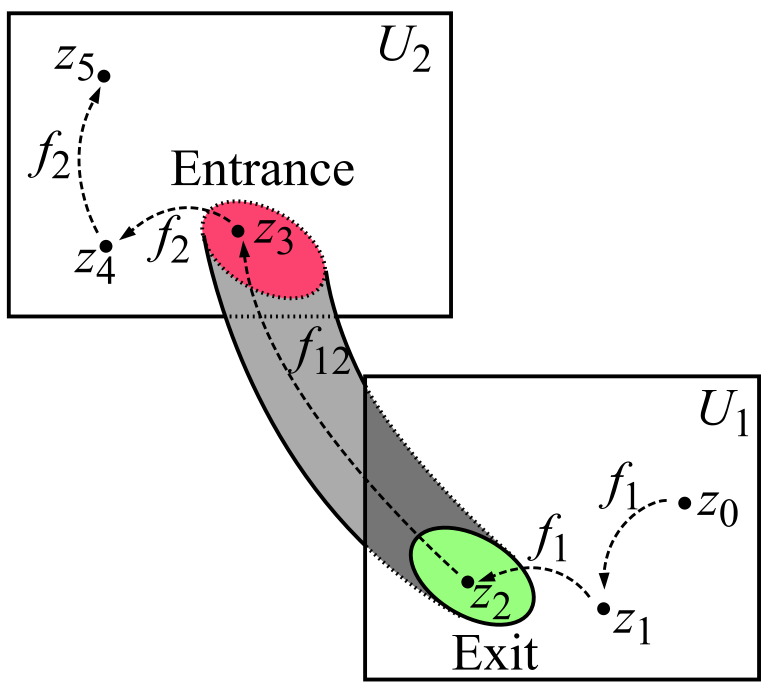




Multi-scale dynamics: Part 1

- Slices of energy surface: Poincaré sections U_i
- Tube dynamics: evolution **between** U_i $\longleftarrow$
- What about evolution **on** U_i?





Motion near saddles

- Near L_1 or L_2, linearized vector field has eigenvalues

$$\pm\lambda \text{ and } \pm i\omega_j,\ j = 2, \ldots, N$$

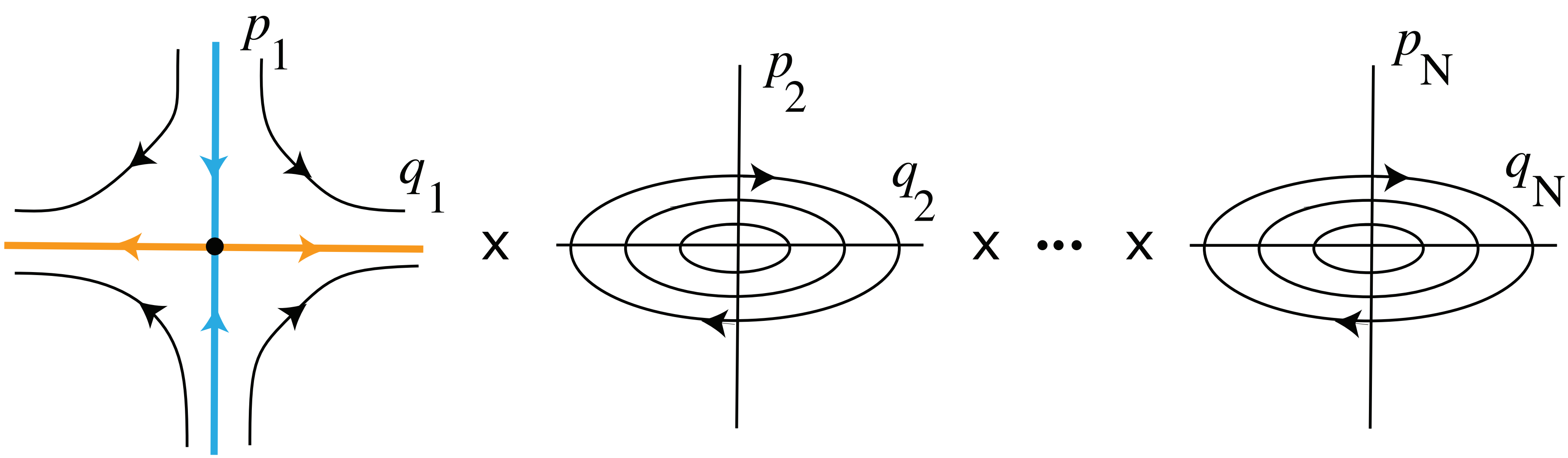
- Under local change of coordinates

$$H(q,p) = \lambda q_1 p_1 + \sum_{i=2}^{N} \frac{\omega_i}{2}\left(p_i^2 + q_i^2\right)$$

Motion near saddles

- Equilibrium point is of type
  saddle $\times$ center $\times \cdots \times$ center $(N-1$ centers)

  i.e., **rank 1 saddle**

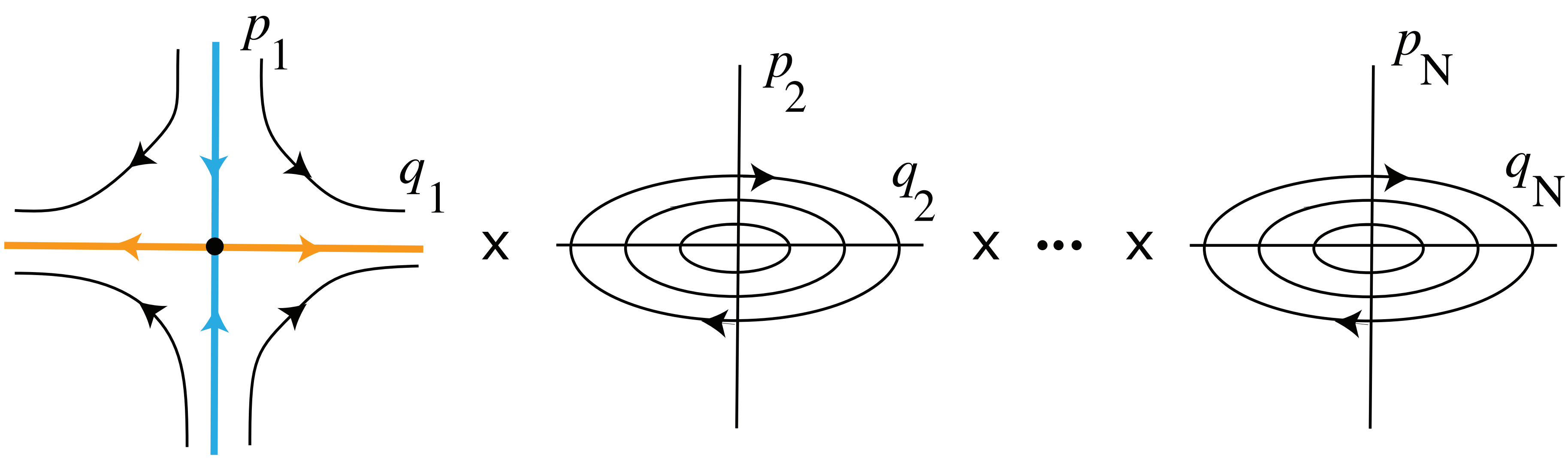


the N canonical planes

Motion near saddles

- For energy h just above saddle pt, $(q_1, p_1) = (0, 0)$ is normally hyperbolic invariant manifold of bound orbits

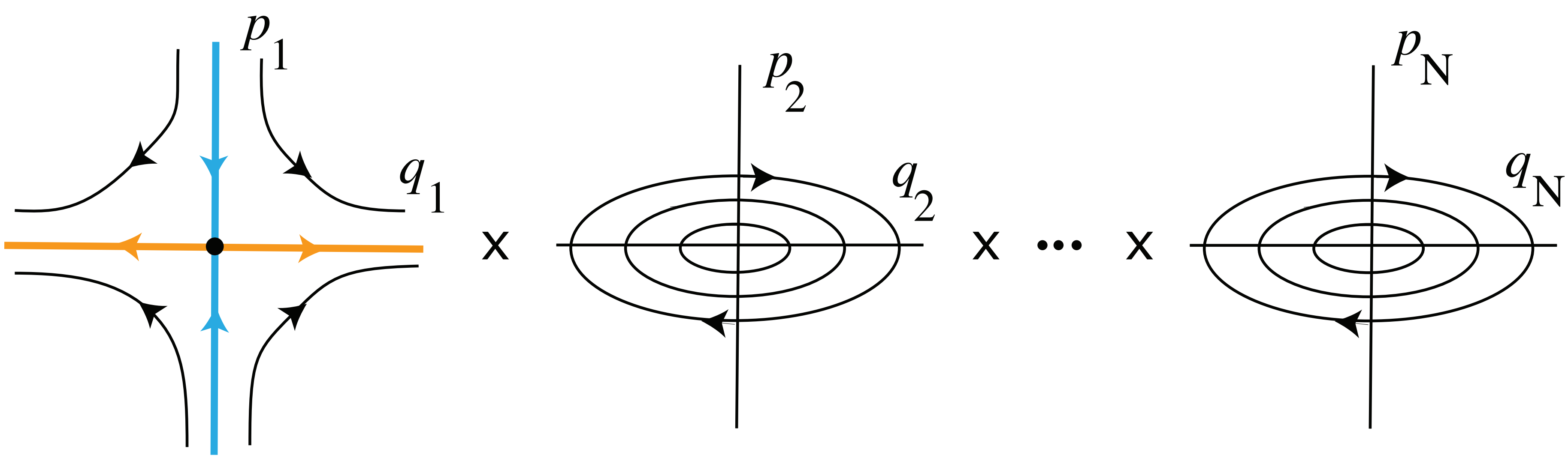
$$\mathcal{M}_h = \sum_{i=2}^{N} \frac{\omega_i}{2} \left(p_i^2 + q_i^2\right) = h > 0.$$



the N canonical planes

Motion near saddles

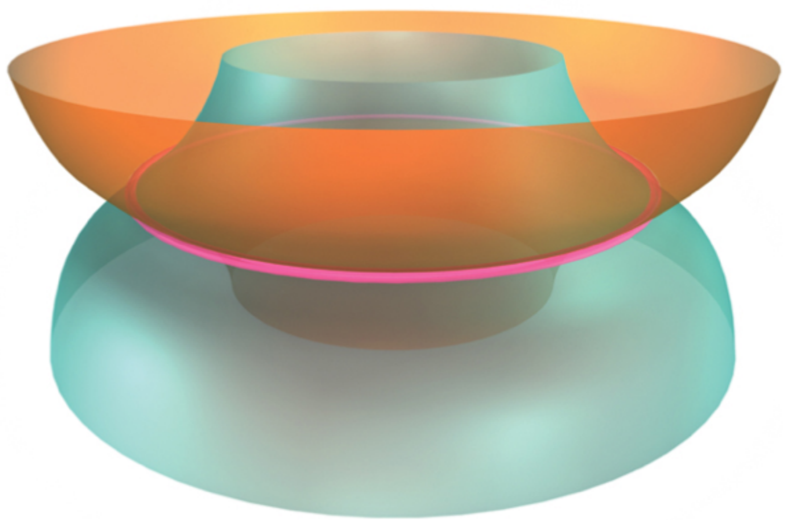
- Note that $\mathcal{M}_h \simeq S^{2N-3}$
  - $N = 2$, the circle S^1, a single periodic orbit
  - $N = 3$, the 3-sphere S^3, a set of periodic and quasi-periodic orbits

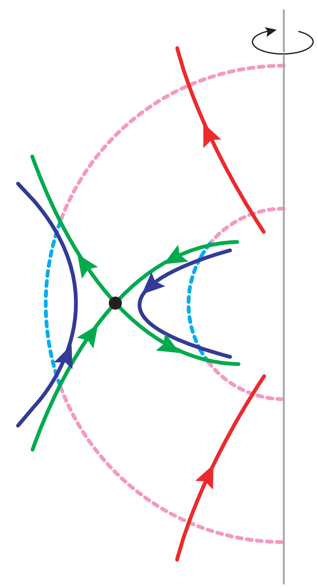


the N canonical planes

Motion near saddles

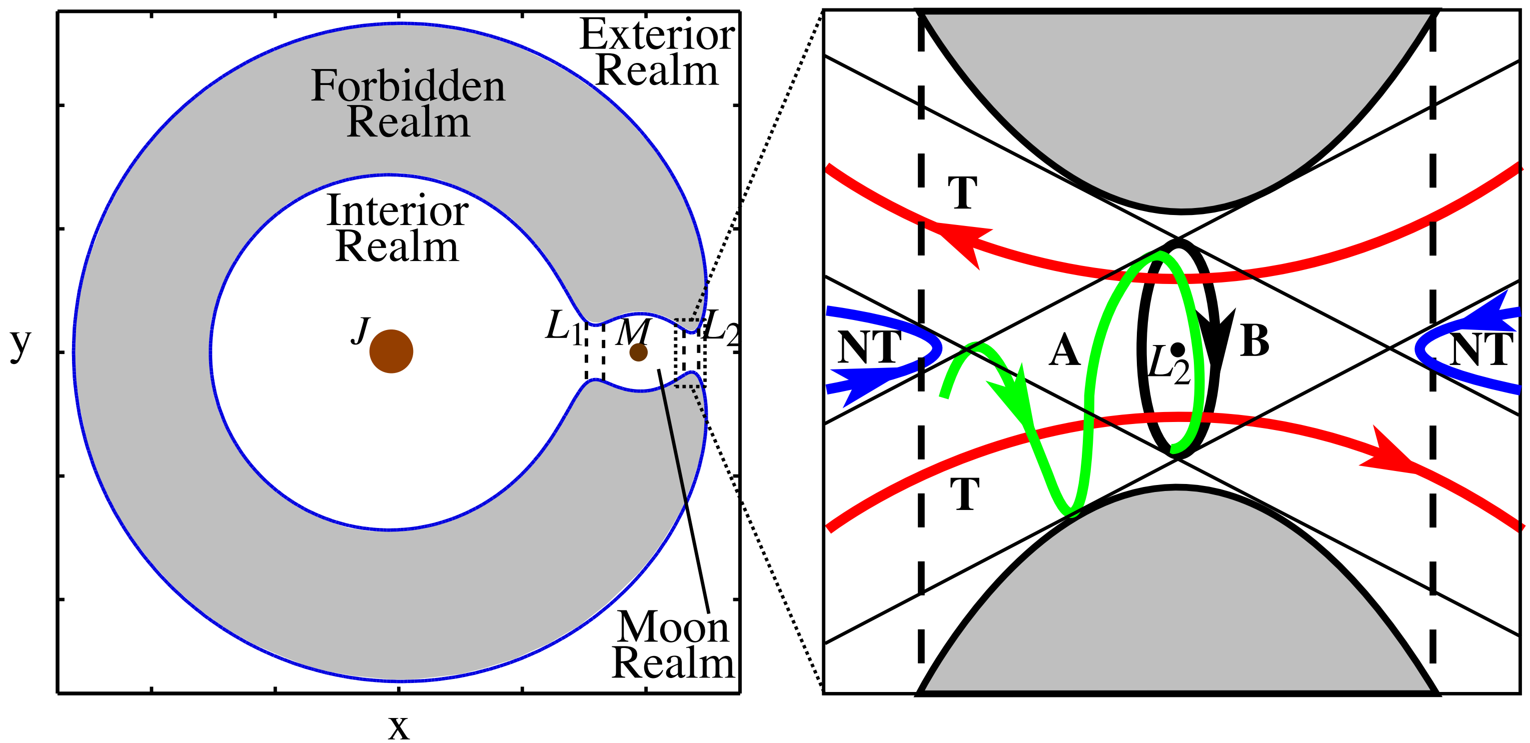
- Note that $\mathcal{M}_h \simeq S^{2N-3}$
  - $N = 2$, the circle S^1, a single periodic orbit
  - $N = 3$, the 3-sphere S^3, a set of periodic and quasi-periodic orbits
- Four "cylinders" or **tubes** of asymptotic orbits: stable, unstable manifolds, $W^s_\pm(\mathcal{M}_h), W^u_\pm(\mathcal{M}_h)$, $\simeq S^3 \times \mathbb{R}$.





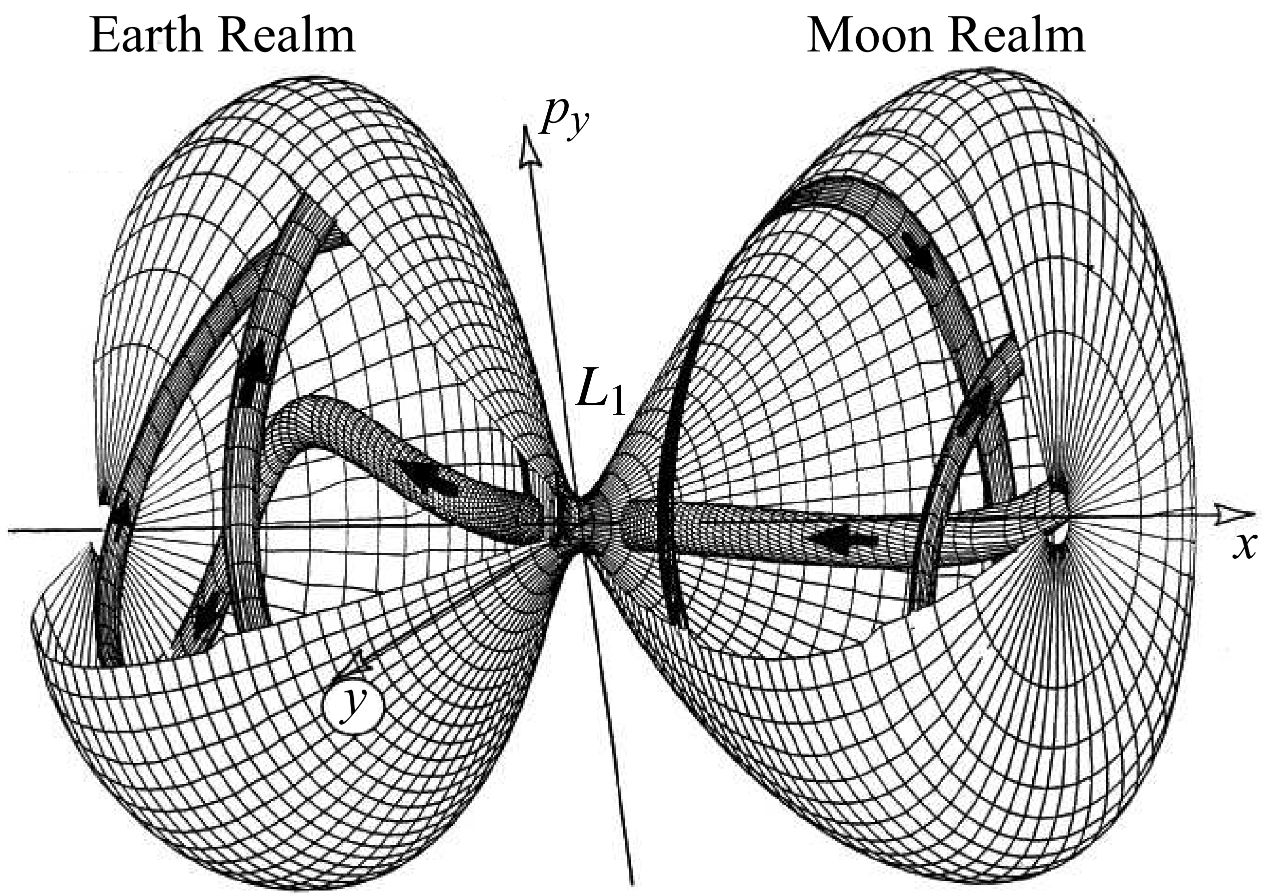
Motion near saddles: 3-body problem

- **B** : **bounded orbits** (periodic/quasi-periodic): S^3
- **A** : **asymptotic orbits** to 3-sphere: $S^3 \times \mathbb{R}$ **(tubes)**
- **T** : **transit** and **NT** : **non-transit** orbits.



Projection to configuration space.

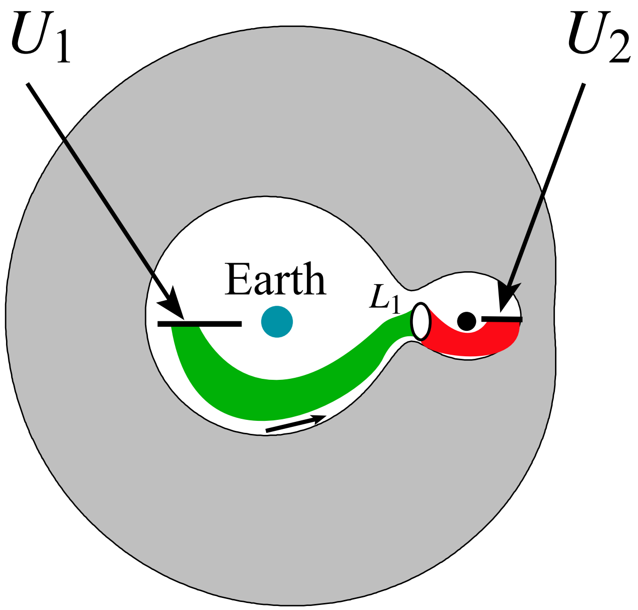
Tube dynamics: inter-realm transport



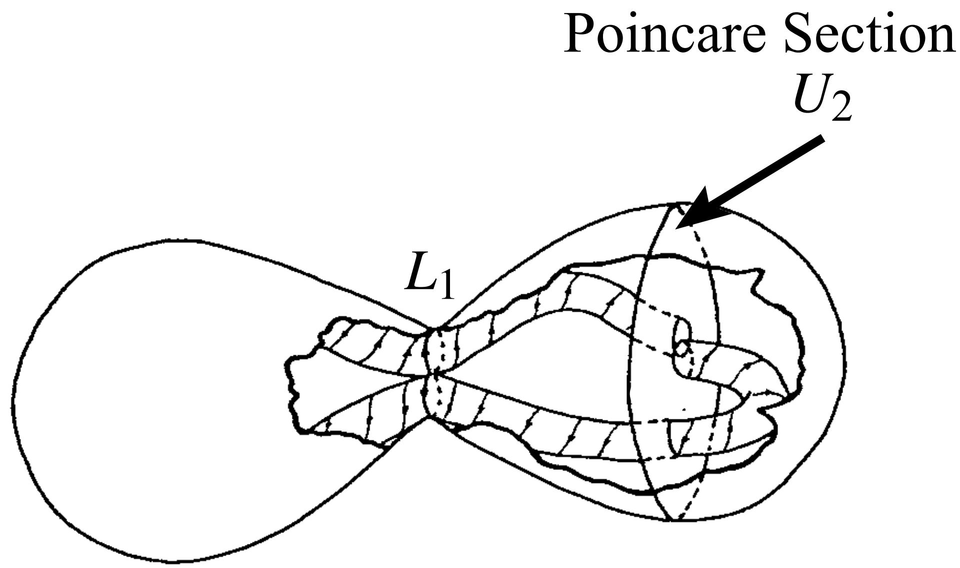
- **Tube dynamics:** All motion between realms connected by necks around saddles must occur through the interior of tubes[1]

[1]Koon, Lo, Marsden, Ross [2000,2001,2002], Gómez, Koon, Lo, Marsden, Masdemont, Ross [2004]

Tube dynamics



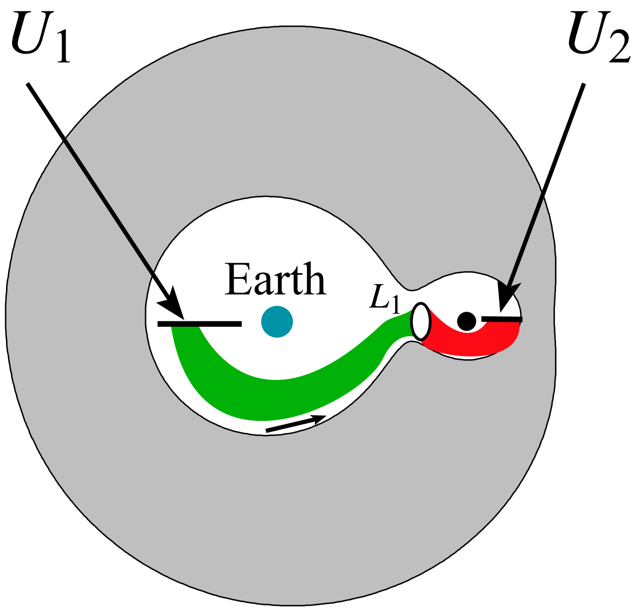
Position Space



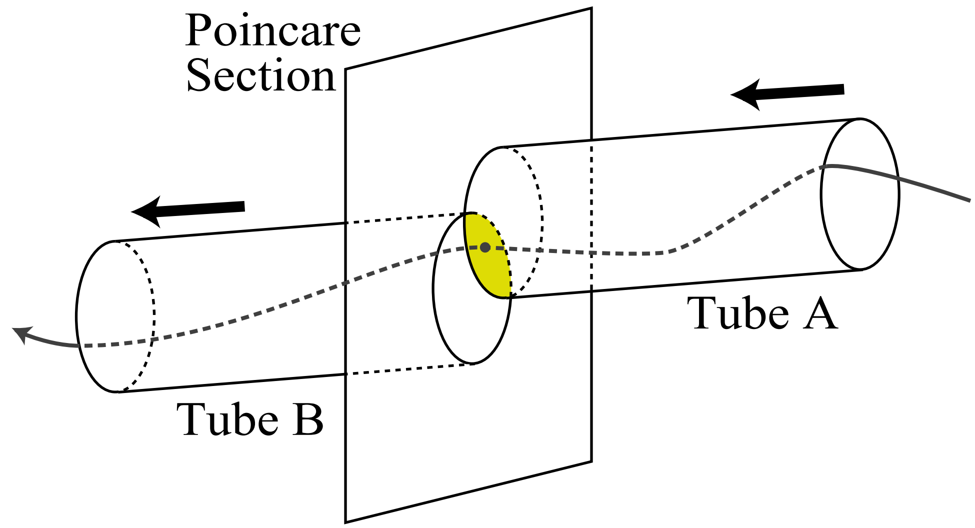
Phase Space

- Motion between Poincaré sections on $\mathcal{M}(E)$
- System reduced to k-map dynamics between the k U_i

Tube dynamics



Position Space

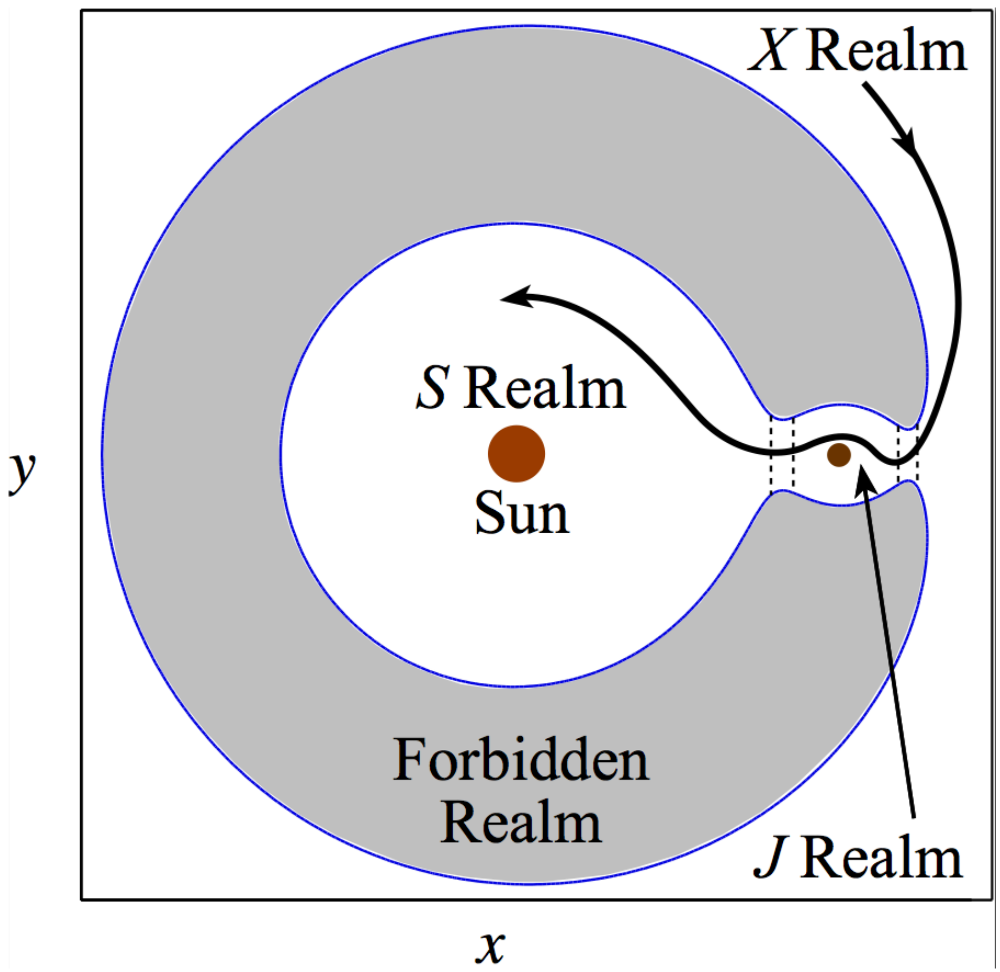


Phase Space

- Motion between Poincaré sections on $\mathcal{M}(E)$
- System reduced to k-map dynamics between the k U_i

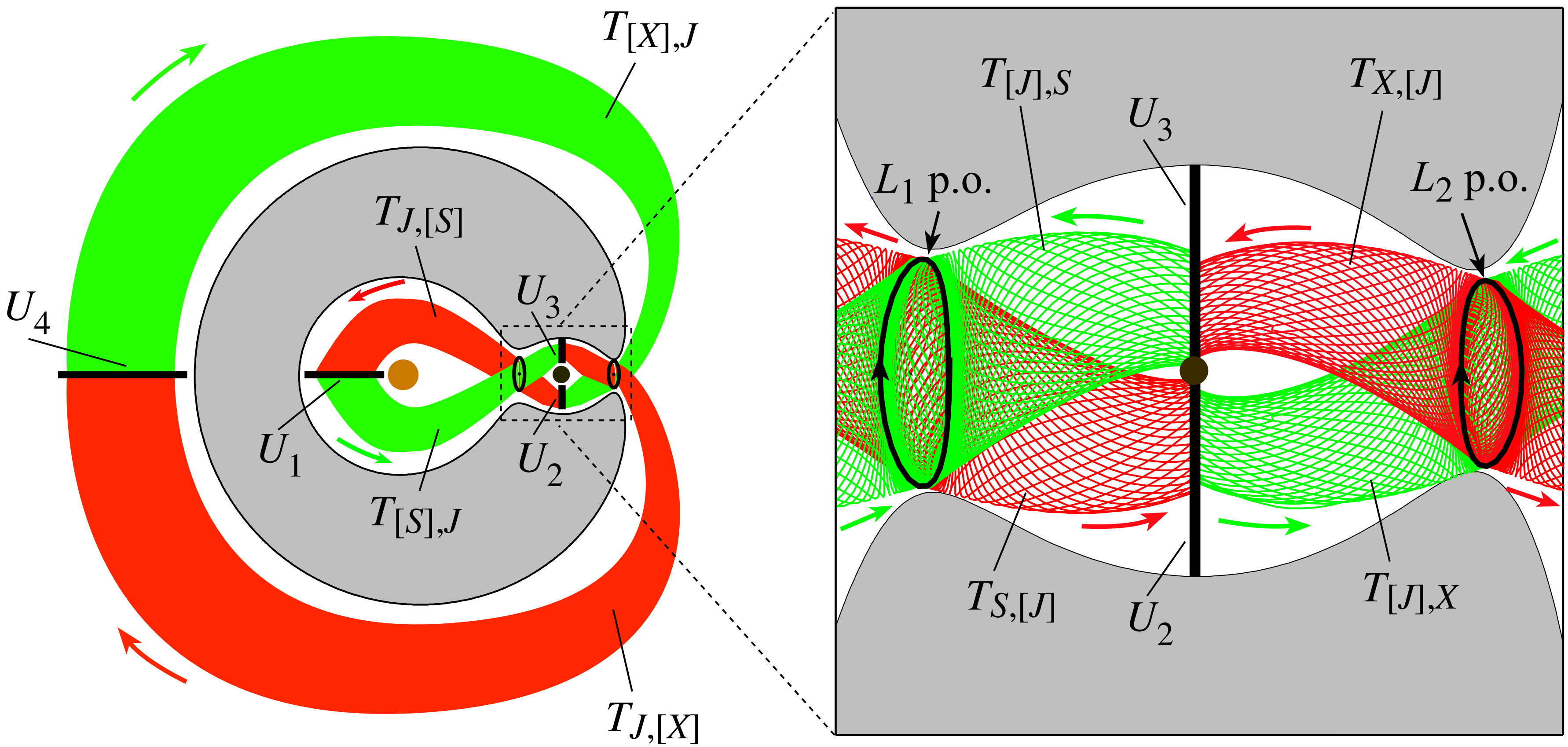
Construction of orbits

- search for an initial condition with a given **itinerary**
- first in 2 d.o.f., then in 3 d.o.f.

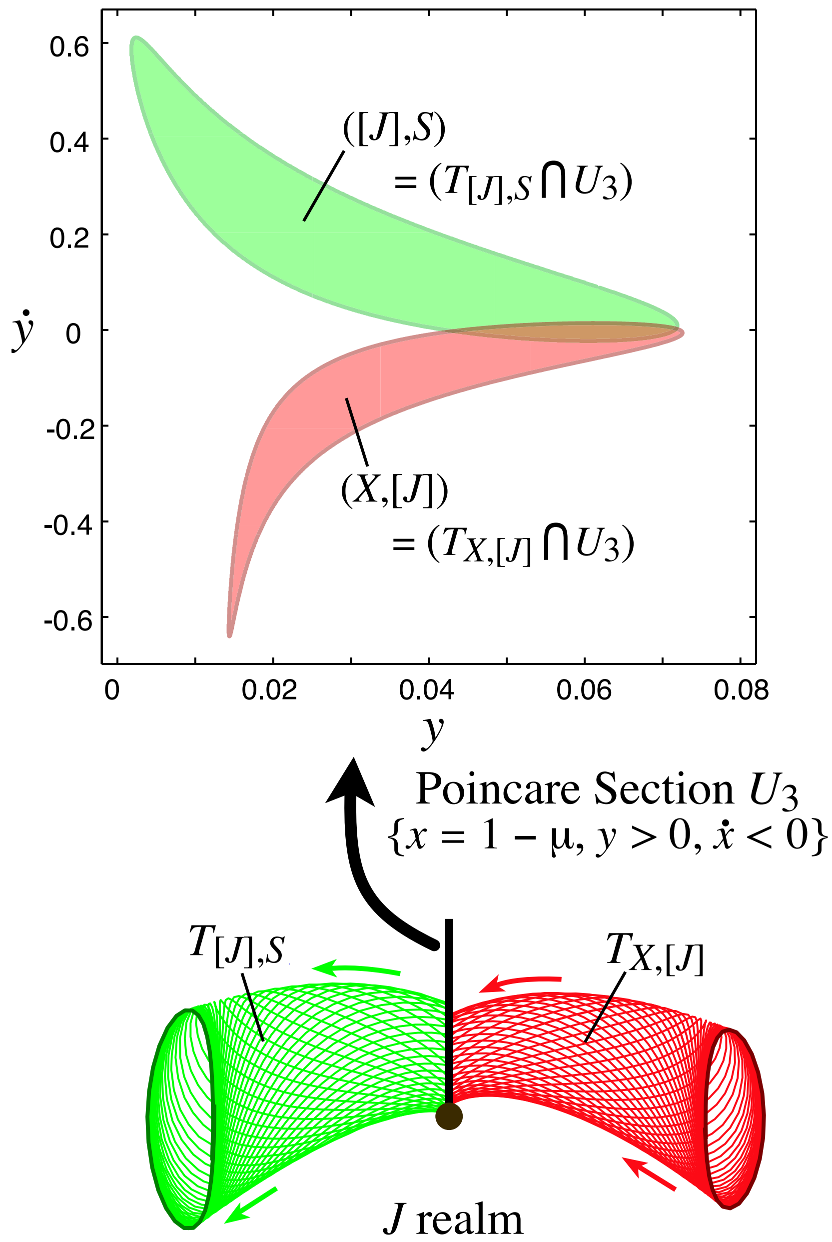


Construction of orbits — 2 d.o.f.

- Consider how tubes connect the U_i

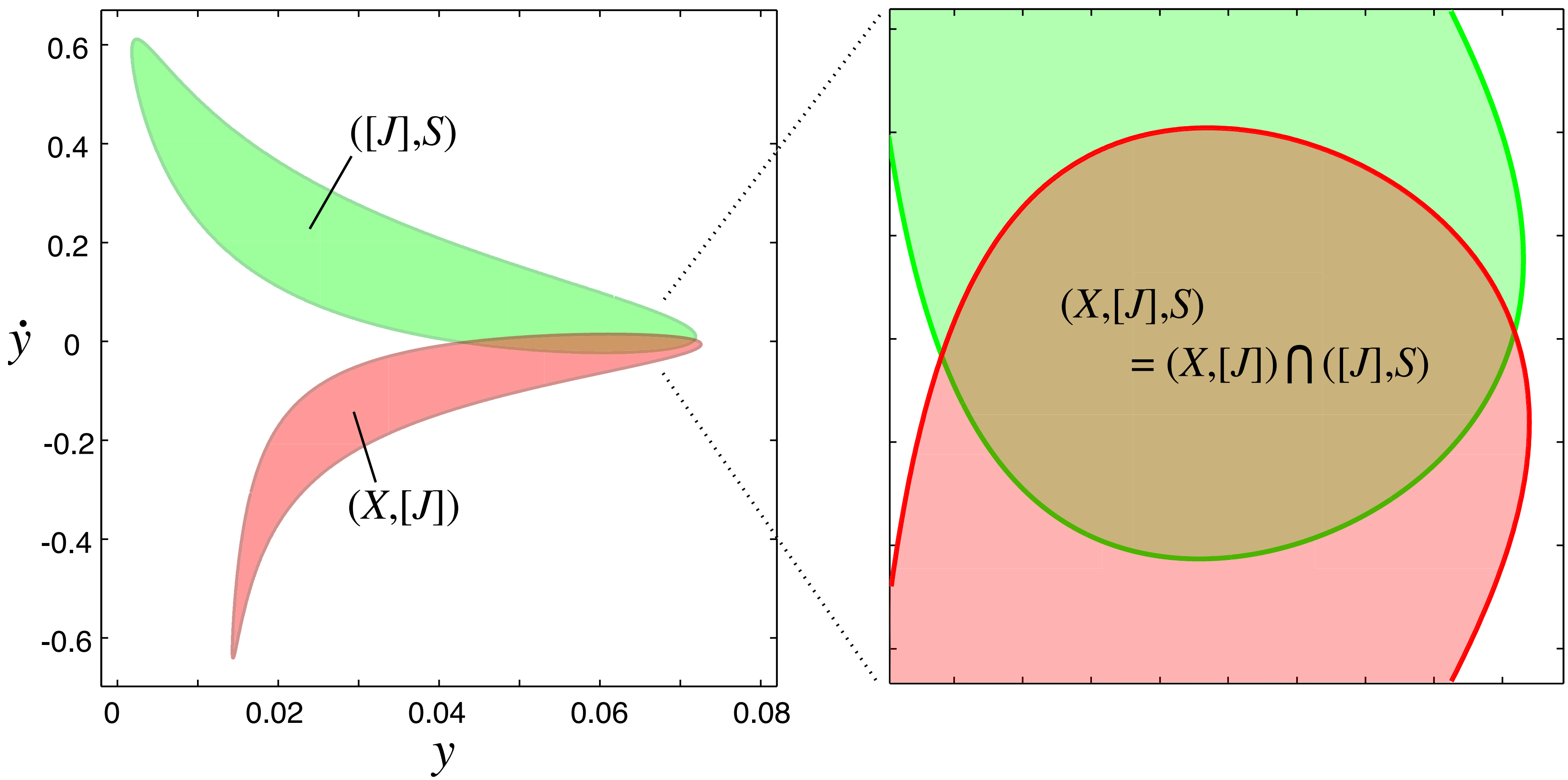


Construction of orbits — 2 d.o.f.



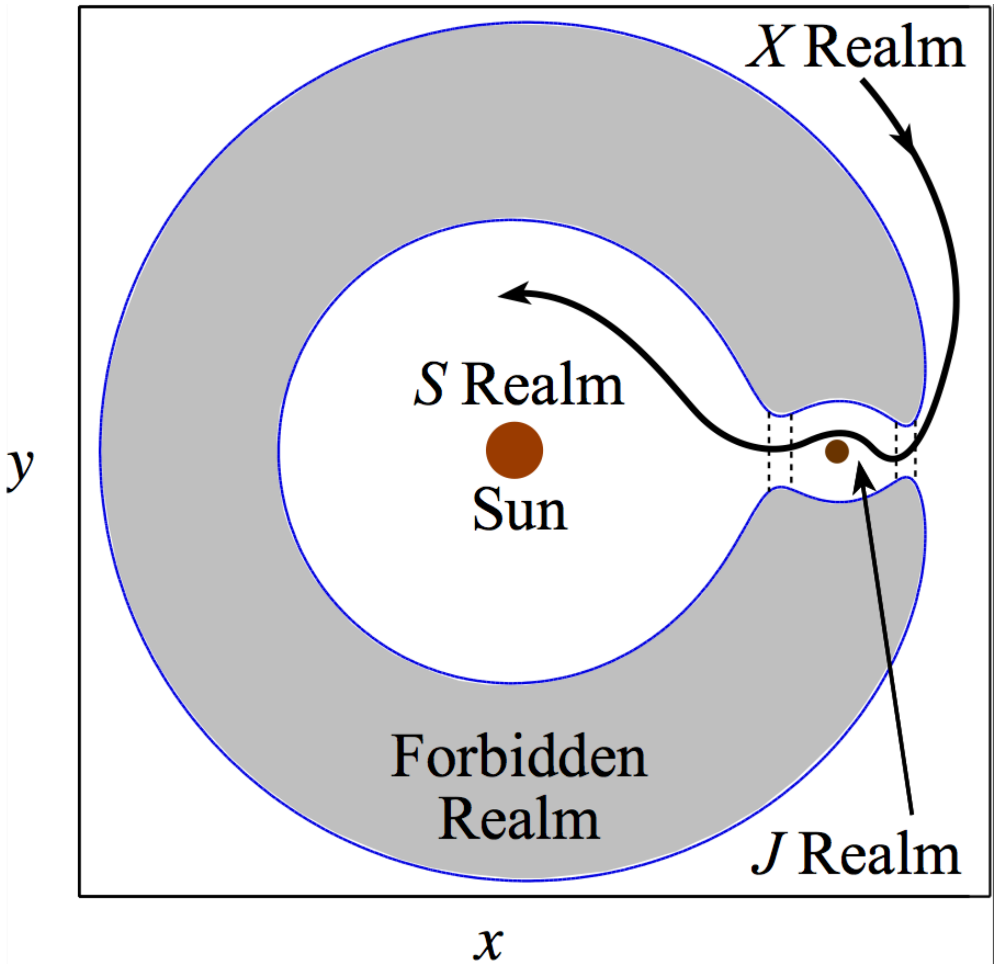
Construction of orbits — 2 d.o.f.

- Denote the intersection $(X,[J])\bigcap([J],S)$ by $(X,[J],S)$



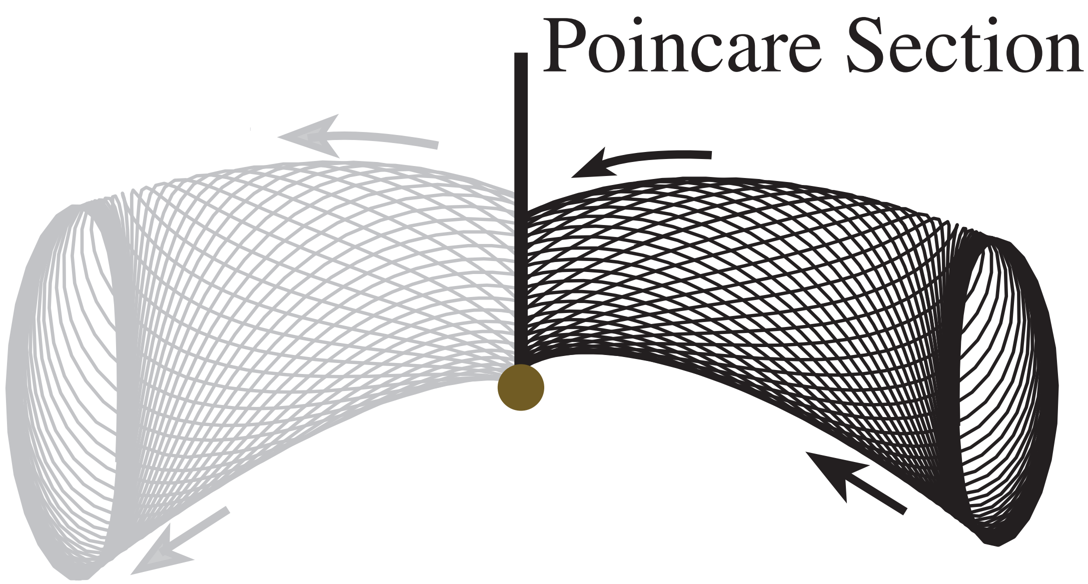
Construction of orbits — 2 d.o.f.

- Forward and backward numerical integration



Construction of orbits — 3 d.o.f.

- **Similar for 3 d.o.f.**: Invariant manifold tubes $S^3 \times \mathbb{R}$
- Poincaré section of energy surface
  - at $x =$ constant, $(y, \dot{y}, z, \dot{z}) \subset \mathbb{R}^4$



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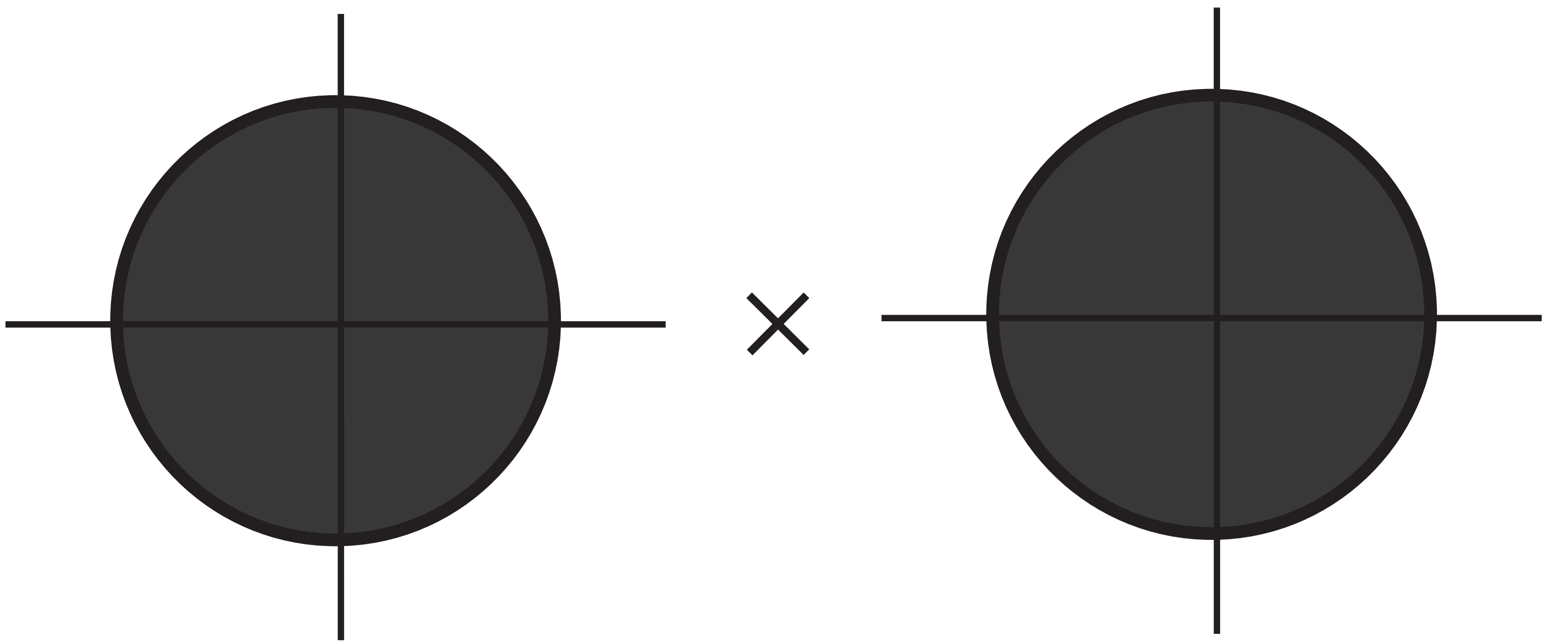
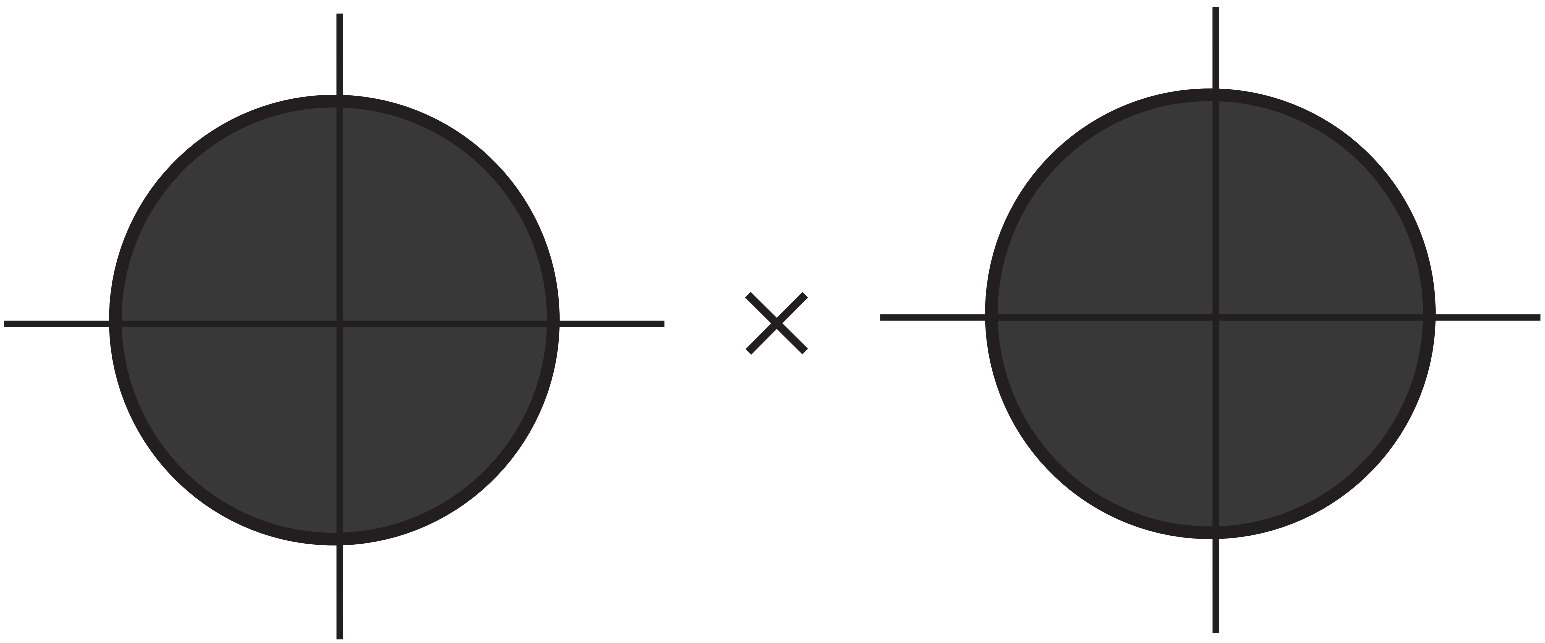
- Poincaré section of energy surface
  - at $x =$ constant, $(y, \dot{y}, z, \dot{z}) \subset \mathbb{R}^4$
- Tube **cross-section** is a topological **3-sphere** S^3 of radius r
  - S^3 projection: **disk** $\times$ **disk**



Determining interior of S^3

- S^3 projection: **disk** $\times$ **disk**

$$\begin{aligned} y^2 + \dot{y}^2 + z^2 + \dot{z}^2 &= r^2 \\ r_y^2 \quad + \quad r_z^2 \quad &= r^2 \end{aligned}$$



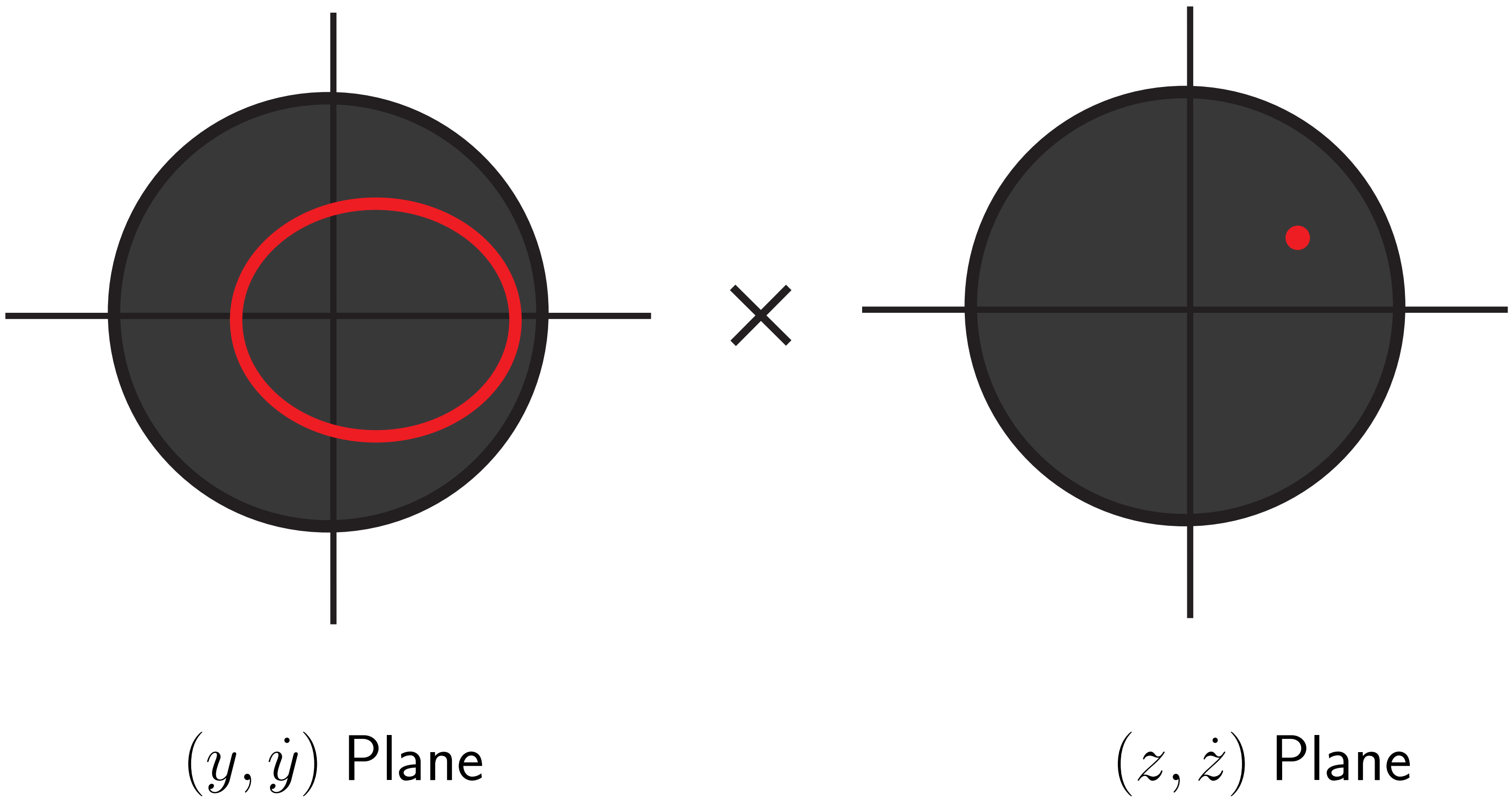
$(y, \dot{y})$ Plane

$(z, \dot{z})$ Plane

Determining interior of S^3

- For fixed $(z, \dot{z})$, projection onto $(y, \dot{y})$ is a **closed curve**

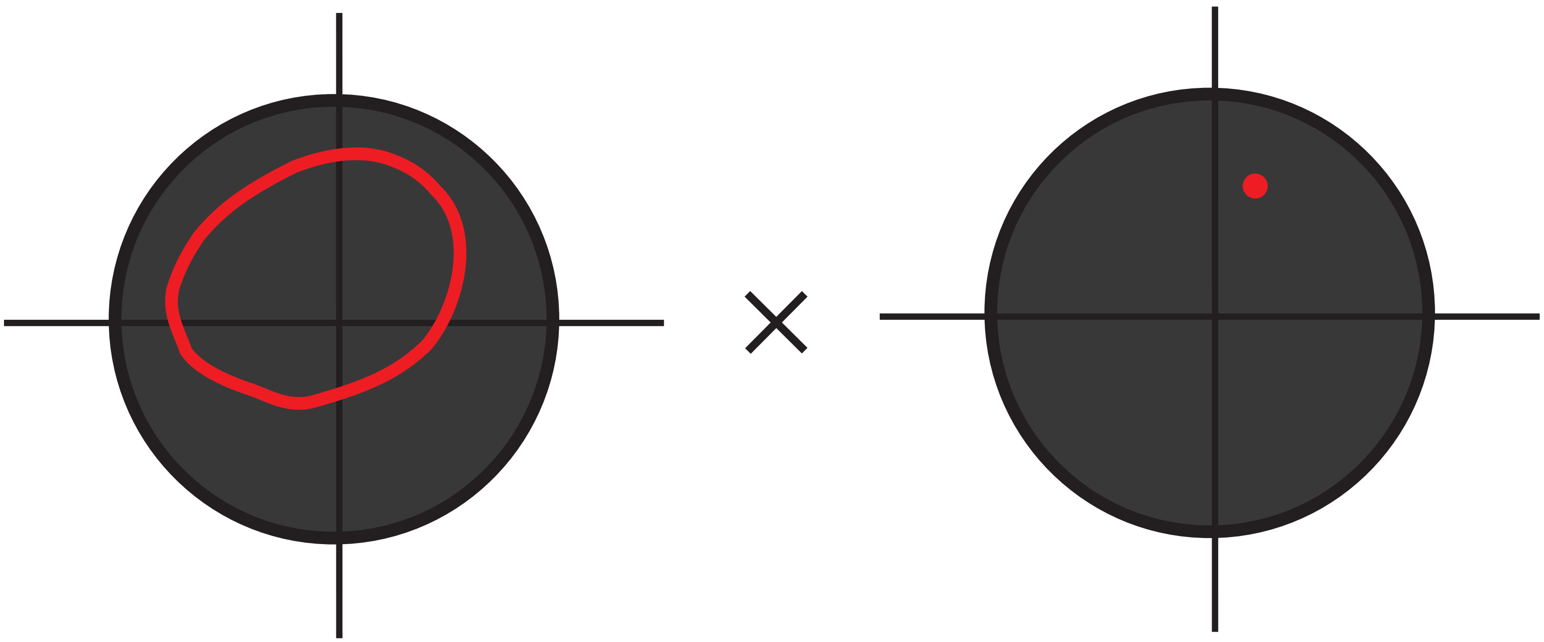
$$\begin{aligned} y^2 + \dot{y}^2 &= r^2 - (z^2 + \dot{z}^2) \\ r_y^2 &= r^2 - r_z^2 \end{aligned}$$



Determining interior of S^3

- For different $(z, \dot{z})$, a different **closed curve** in $(y, \dot{y})$

$$\begin{aligned} y^2 + \dot{y}^2 &= r^2 - (z^2 + \dot{z}^2) \\ r_y^2 &= r^2 - \quad r_z^2 \end{aligned}$$



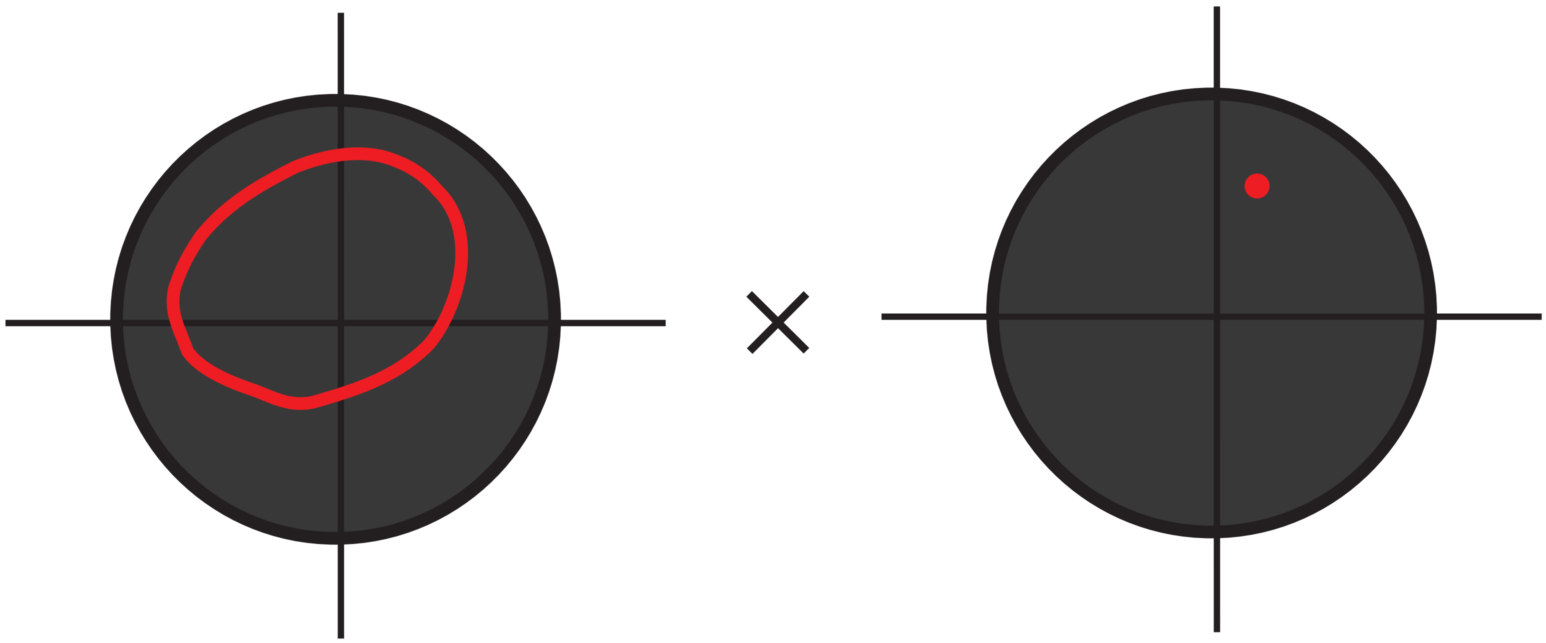
$(y, \dot{y})$ Plane

$(z, \dot{z})$ Plane

Determining interior of S^3

- Cross-section of tube effectively reduced to a **two-parameter family of closed curves**

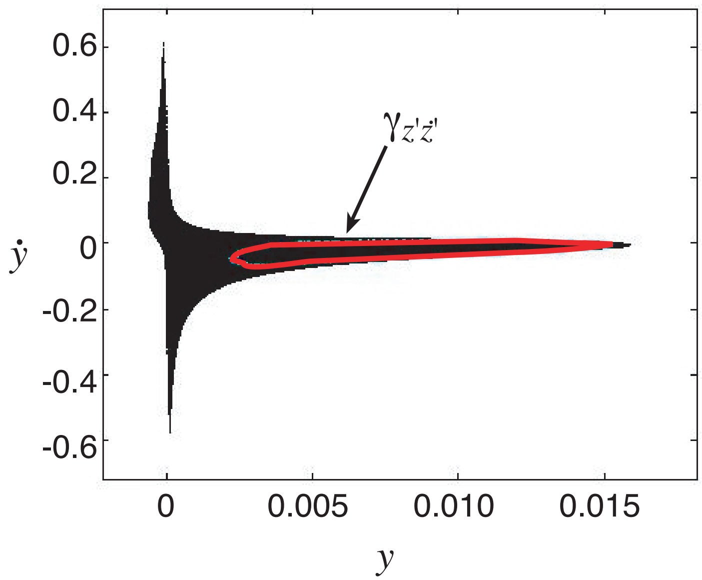
$$y^2 + \dot{y}^2 = r^2 - (z^2 + \dot{z}^2)$$



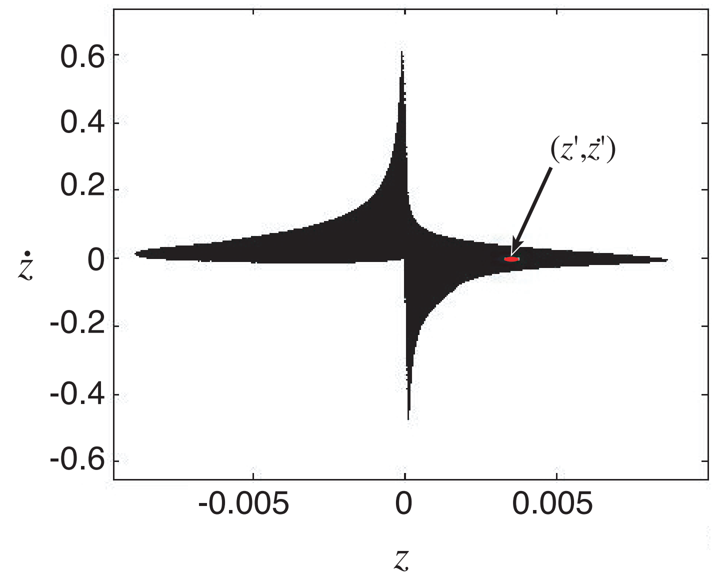
$(y, \dot{y})$ Plane $\times$ $(z, \dot{z})$ Plane

Determining interior of S^3

- Can be demonstrated numerically: $\{\mathrm{int}(\gamma_{z\dot{z}})\}_{(z,\dot{z})}$



$(y, \dot{y})$ Plane

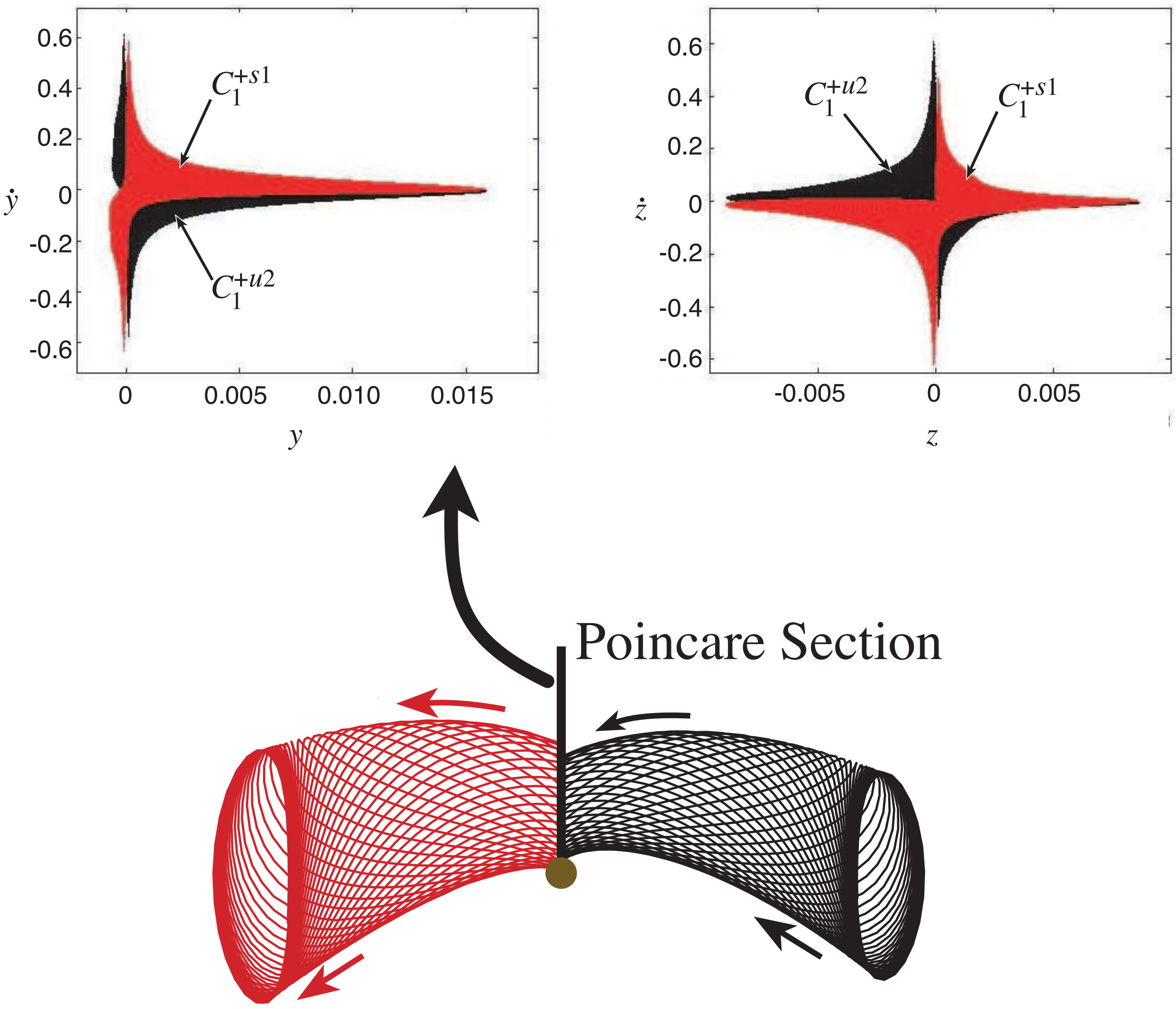


$(z, \dot{z})$ Plane

- Provides nice way to calculate interior of tube, intersections of tubes, etc.

Intersection of phase volumes

- Find (X,J,S) orbit via tube intersection



Intersection of phase volumes

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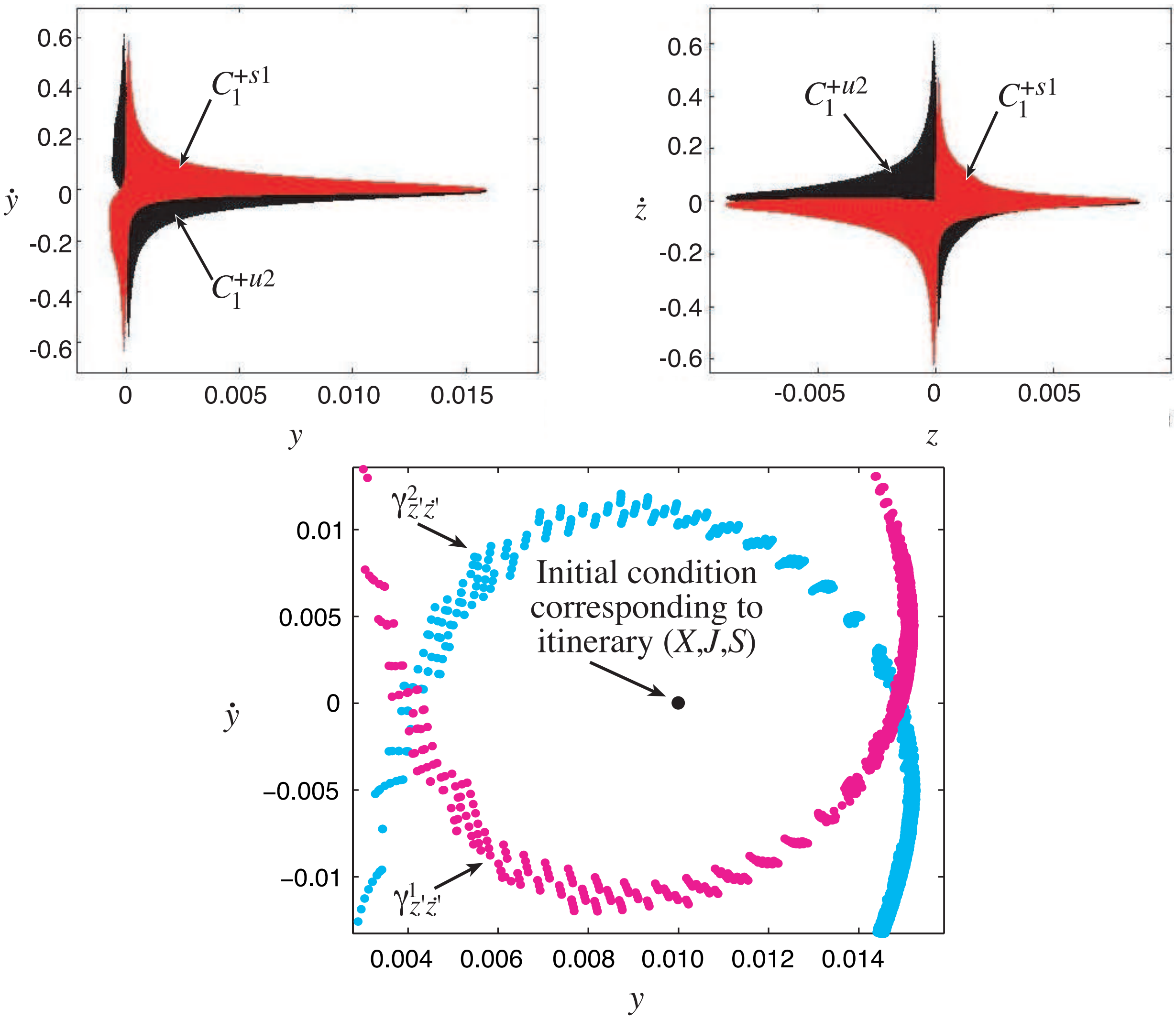
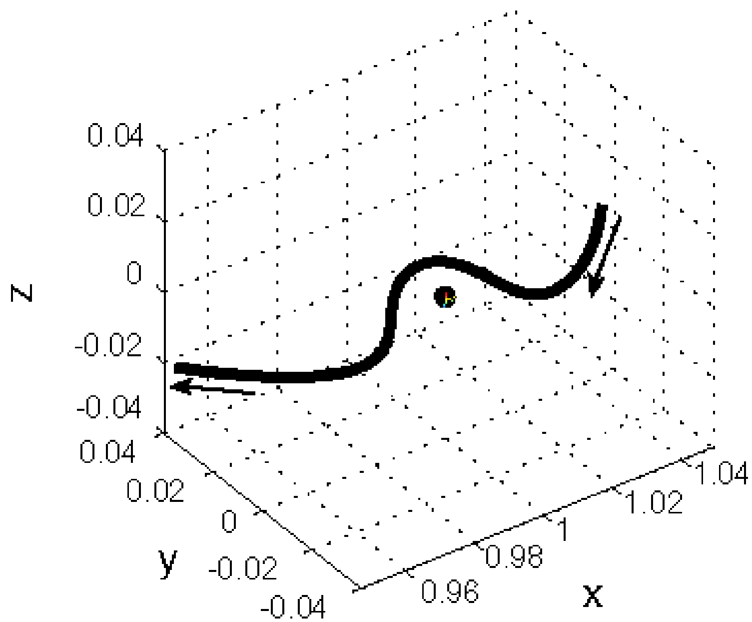
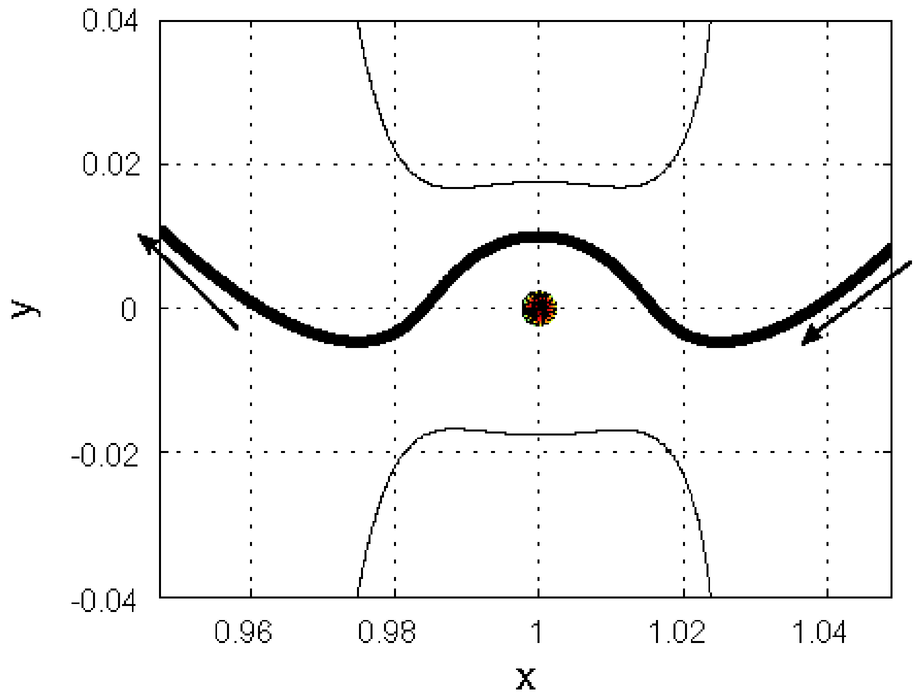


All orbits in intersection correspond to transition



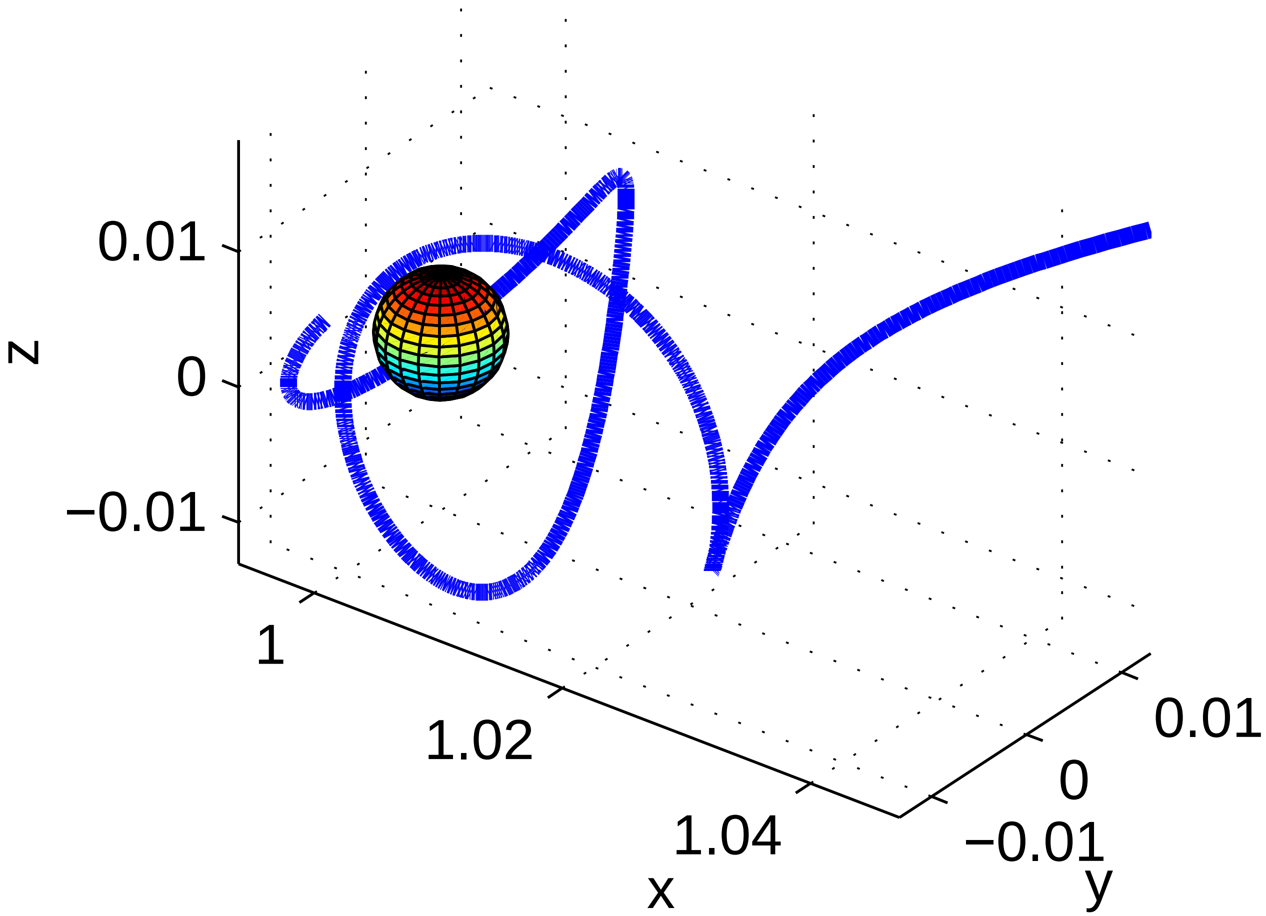
3D view



xy-plane projection

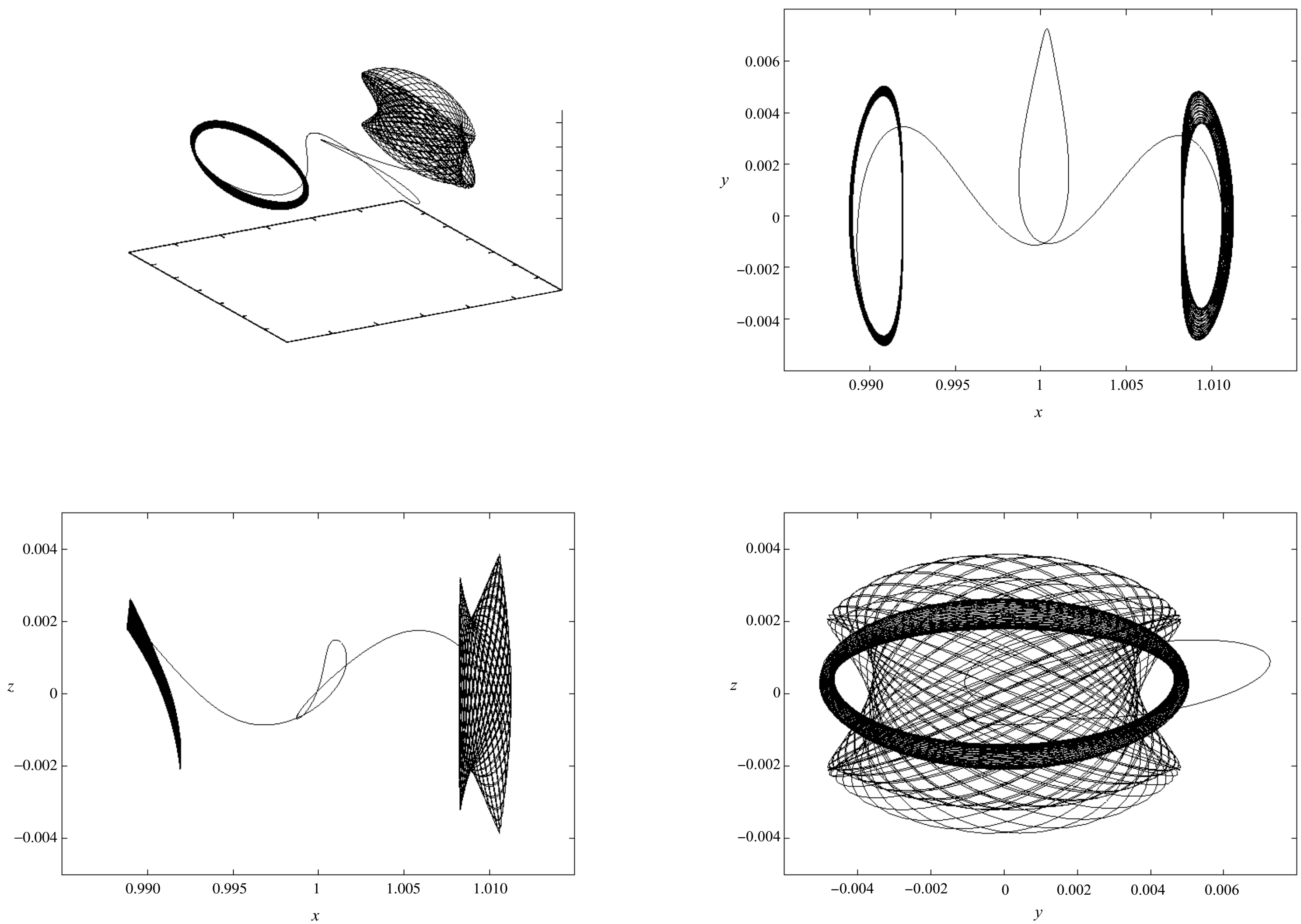
Gómez, Koon, Lo, Marsden, Masdemont, Ross, Nonlinearity [2004]

Other orbits obtained this way



Another example

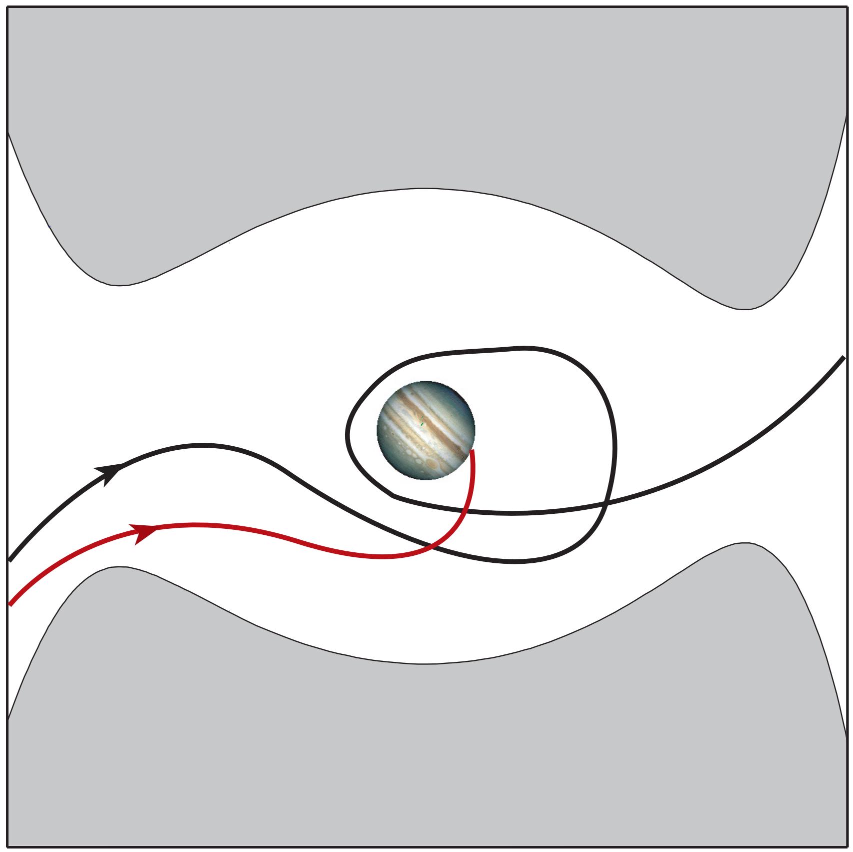
On the tubes, rather than in the tubes



An L_1-L_2 heteroclinic connection

Transition and collision

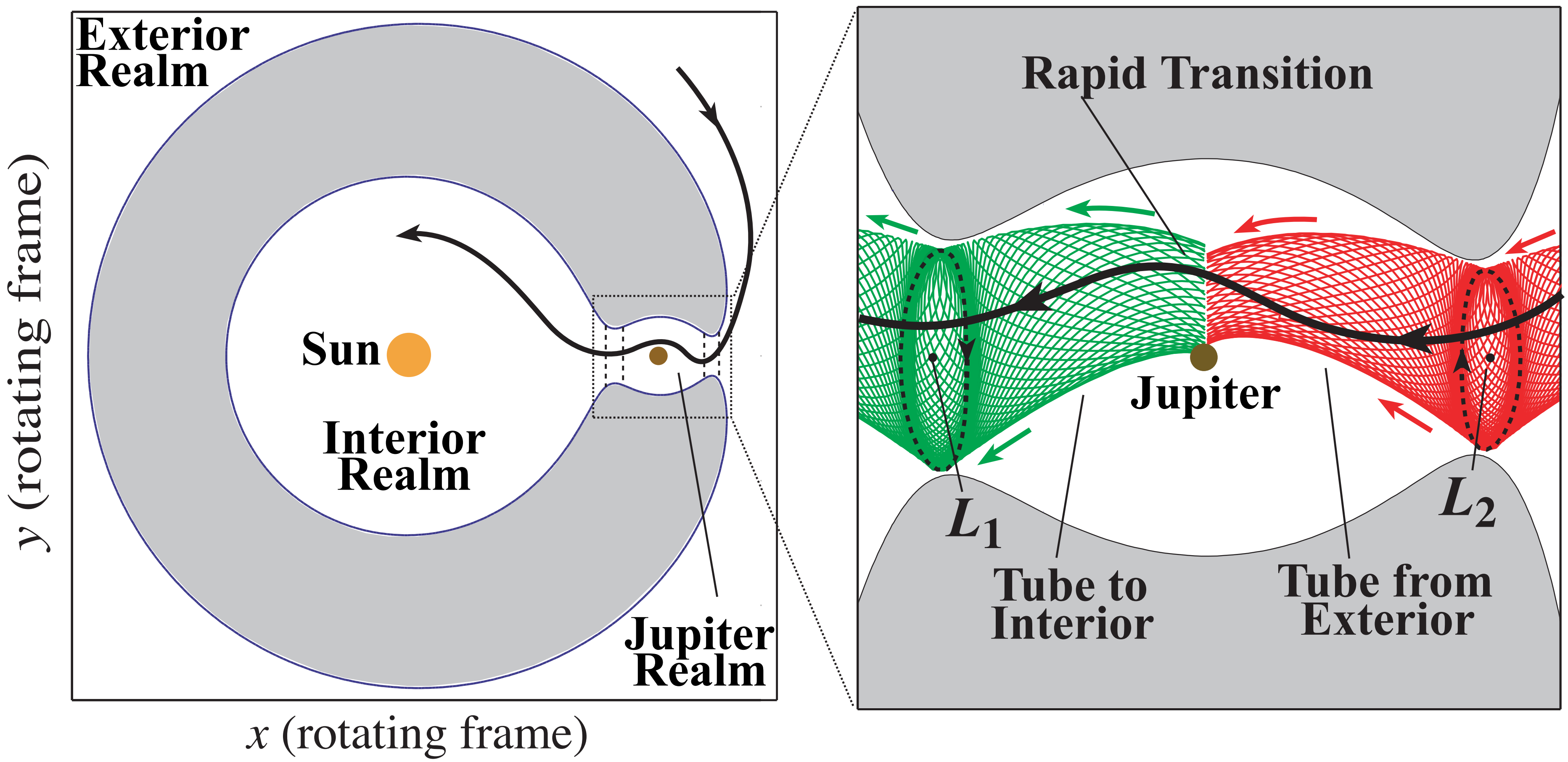
- Interpret relative phase volumes as probabilities[2]



- Transition between realms and/or collision.

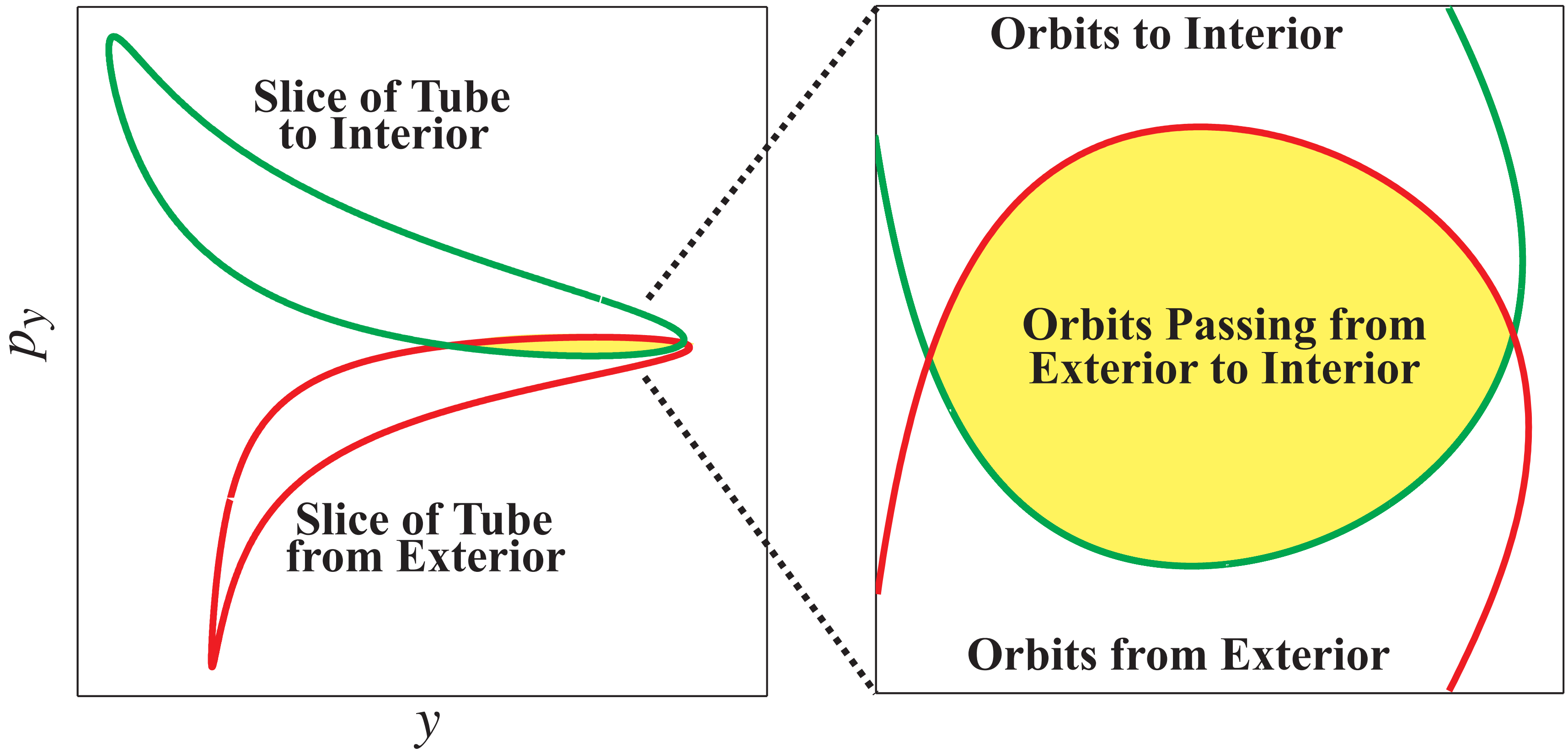
[2]Ross [2003] Statistical theory of interior-exterior transition and collision probabilities for minor bodies in the solar system

Transition probabilities



- Example: Comet transport between outside and inside of Jupiter (i.e., **Oterma**-like transitions)

Transition probabilities

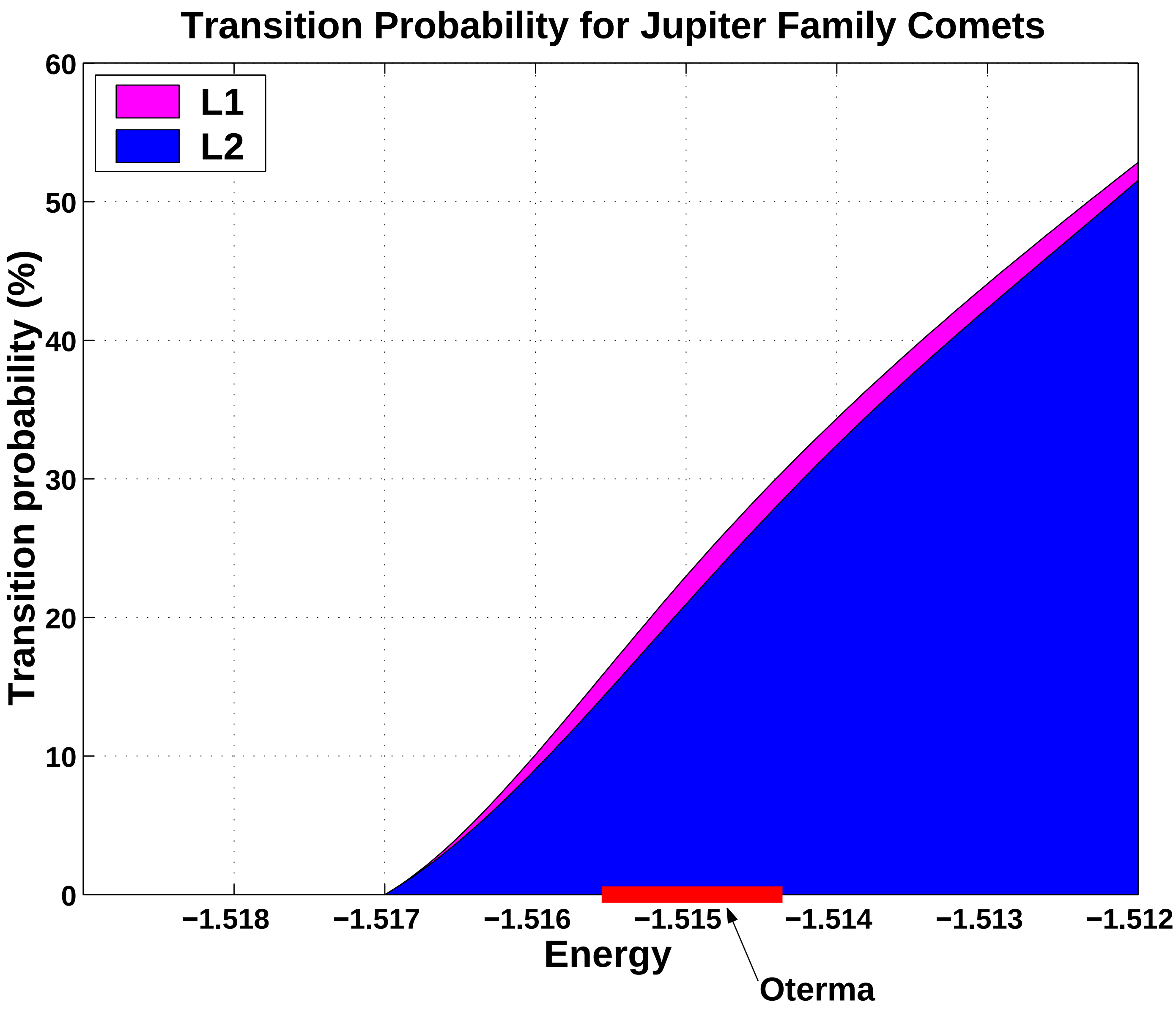


Poincaré Section

- Phase volume ratio gives the **relative probability** to pass from **outside** to **inside** Jupiter's orbit.

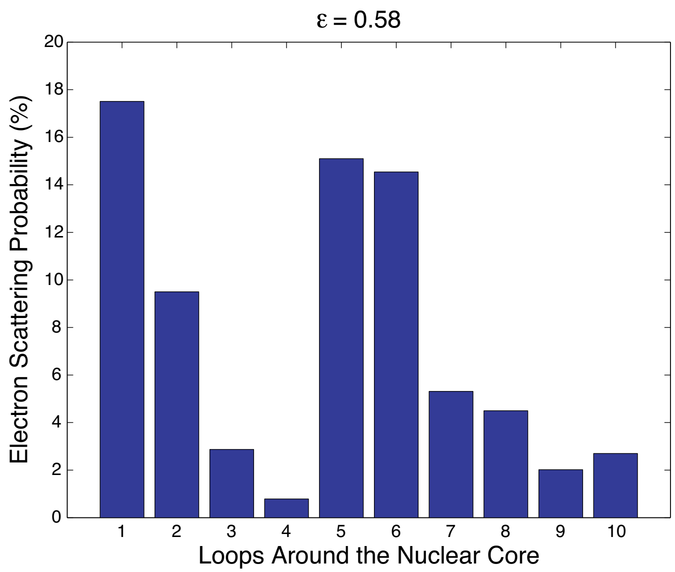
Transition probabilities

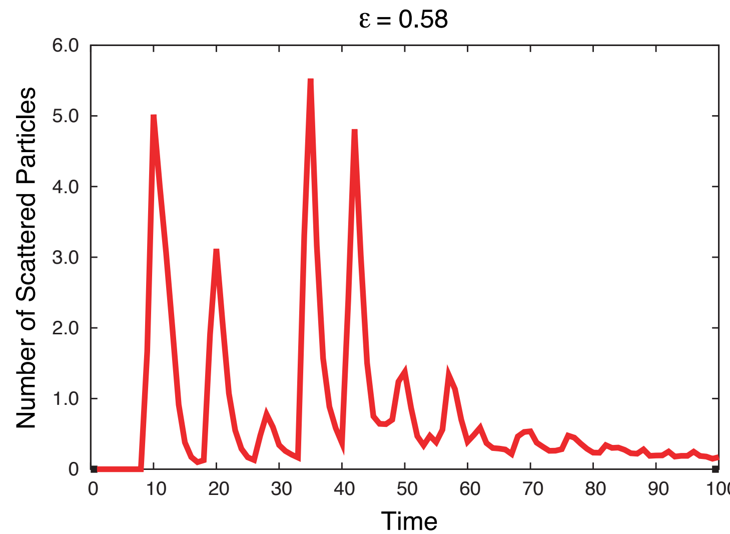
- Jupiter family comet transitions: **X** → **S**, **S** → **X**

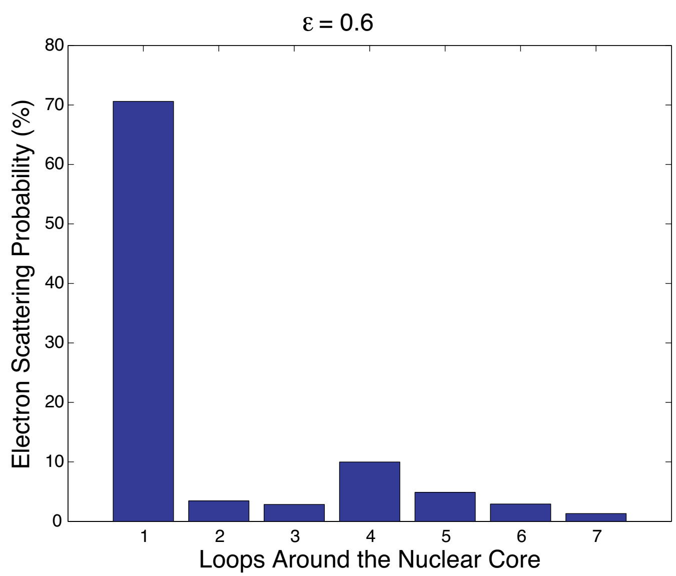


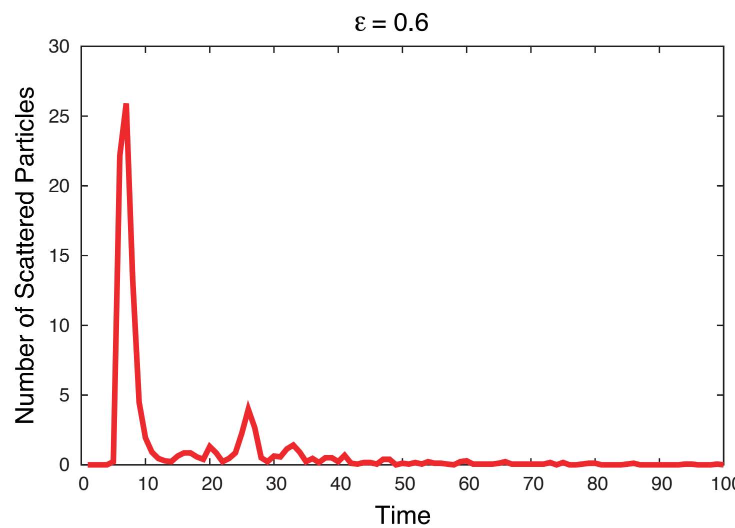
Capture time determined by tube dynamics

- Temporary capture time profiles are structured







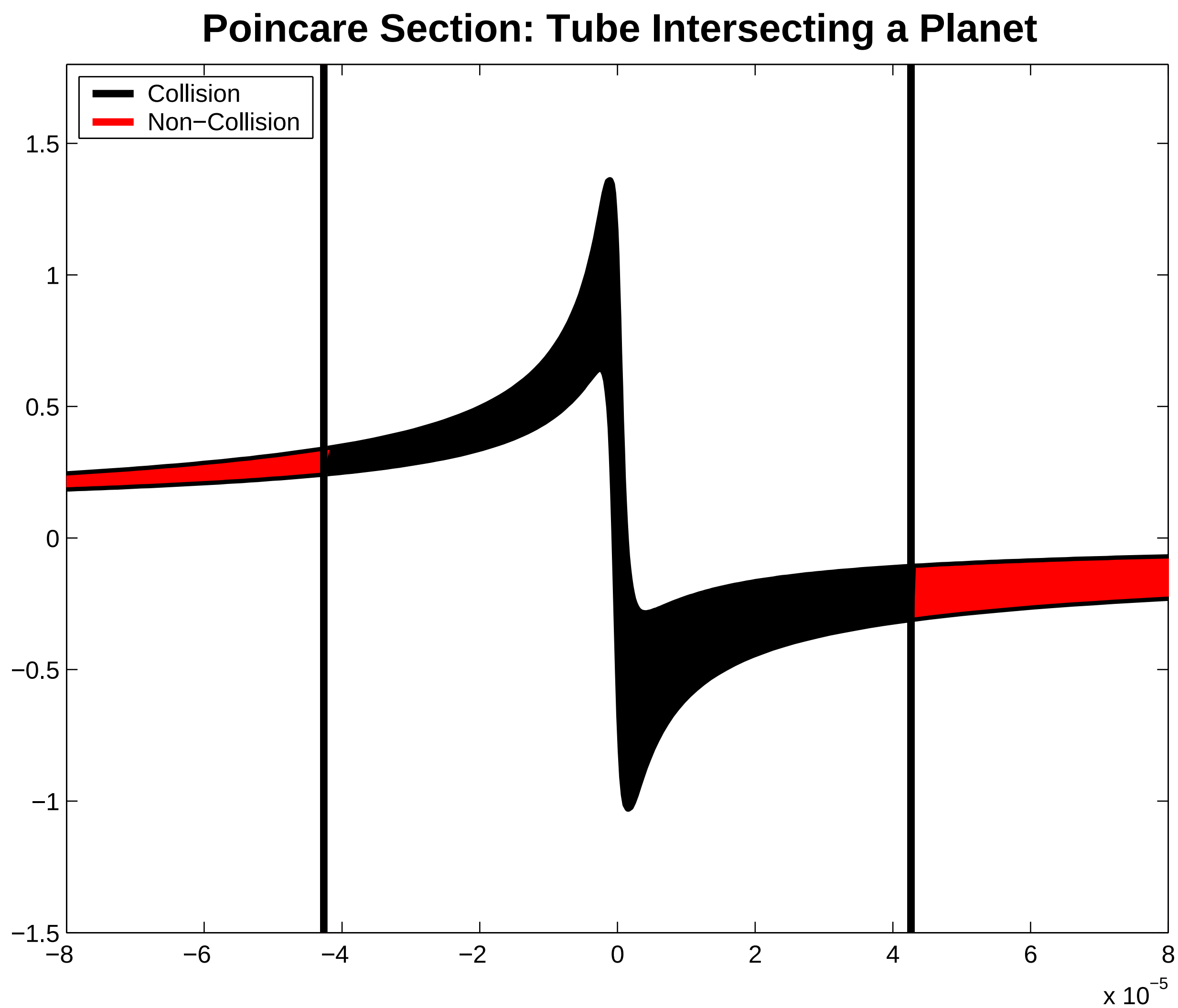


Collision probabilities

- eg, **Shoemaker-Levy 9** and **Earth-impacting asteroids**
- Compute from tube intersection with planet on Poincaré section
- Planetary diameter is a parameter

← Diameter of planet →

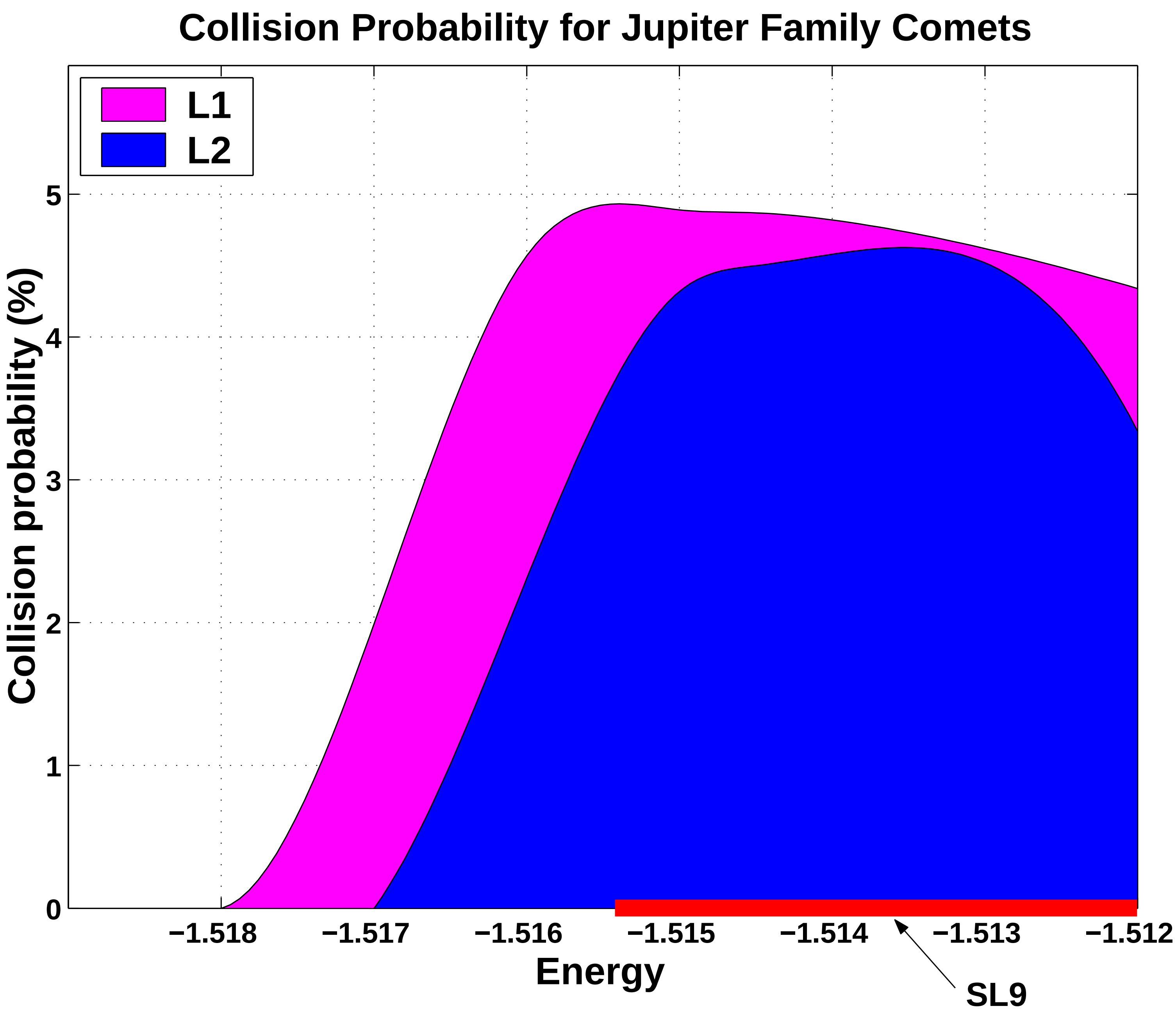
Collision probabilities



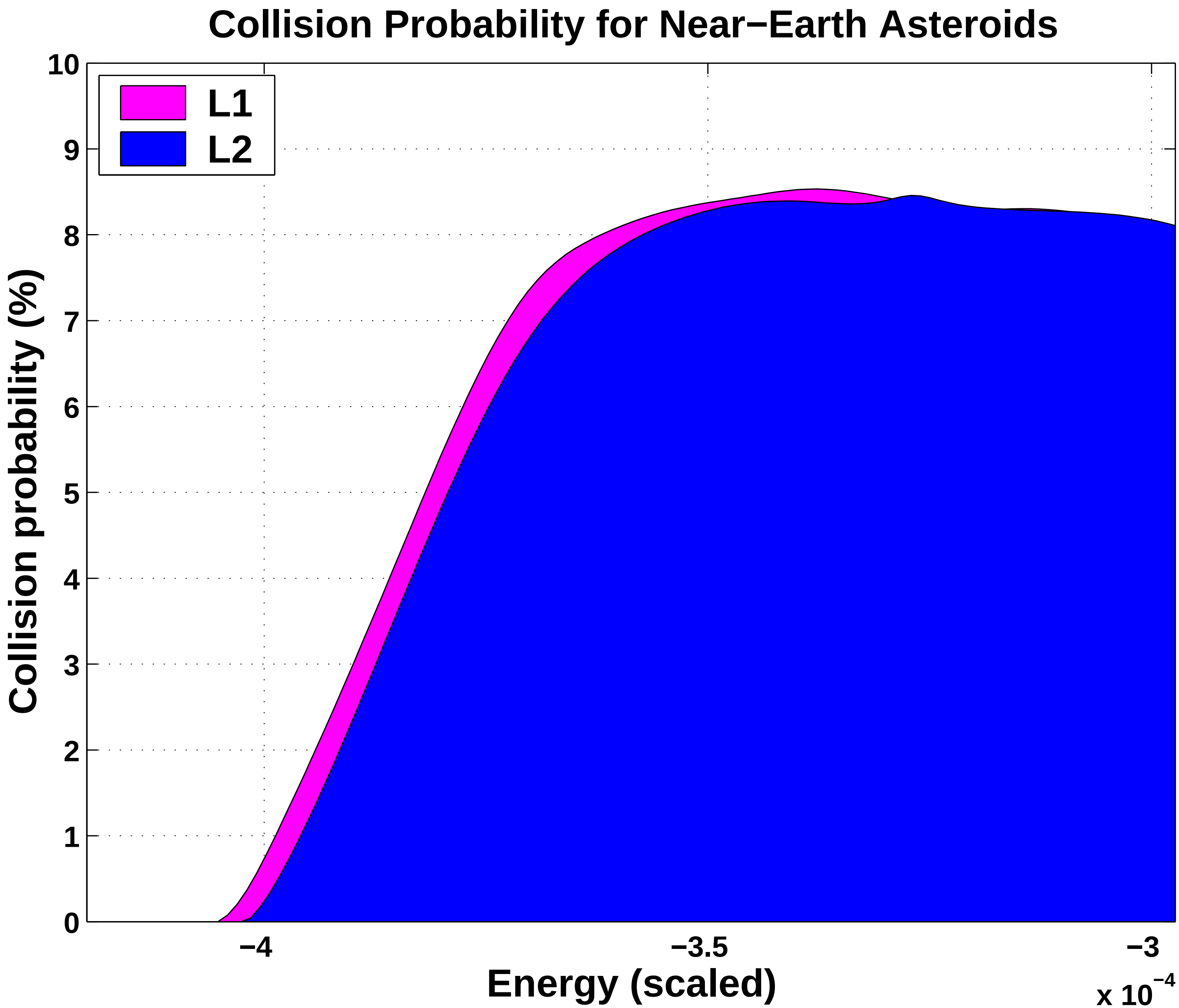
$\longleftarrow$ Diameter of planet $\longrightarrow$

- Poincaré section through planet showing collision portion of tube

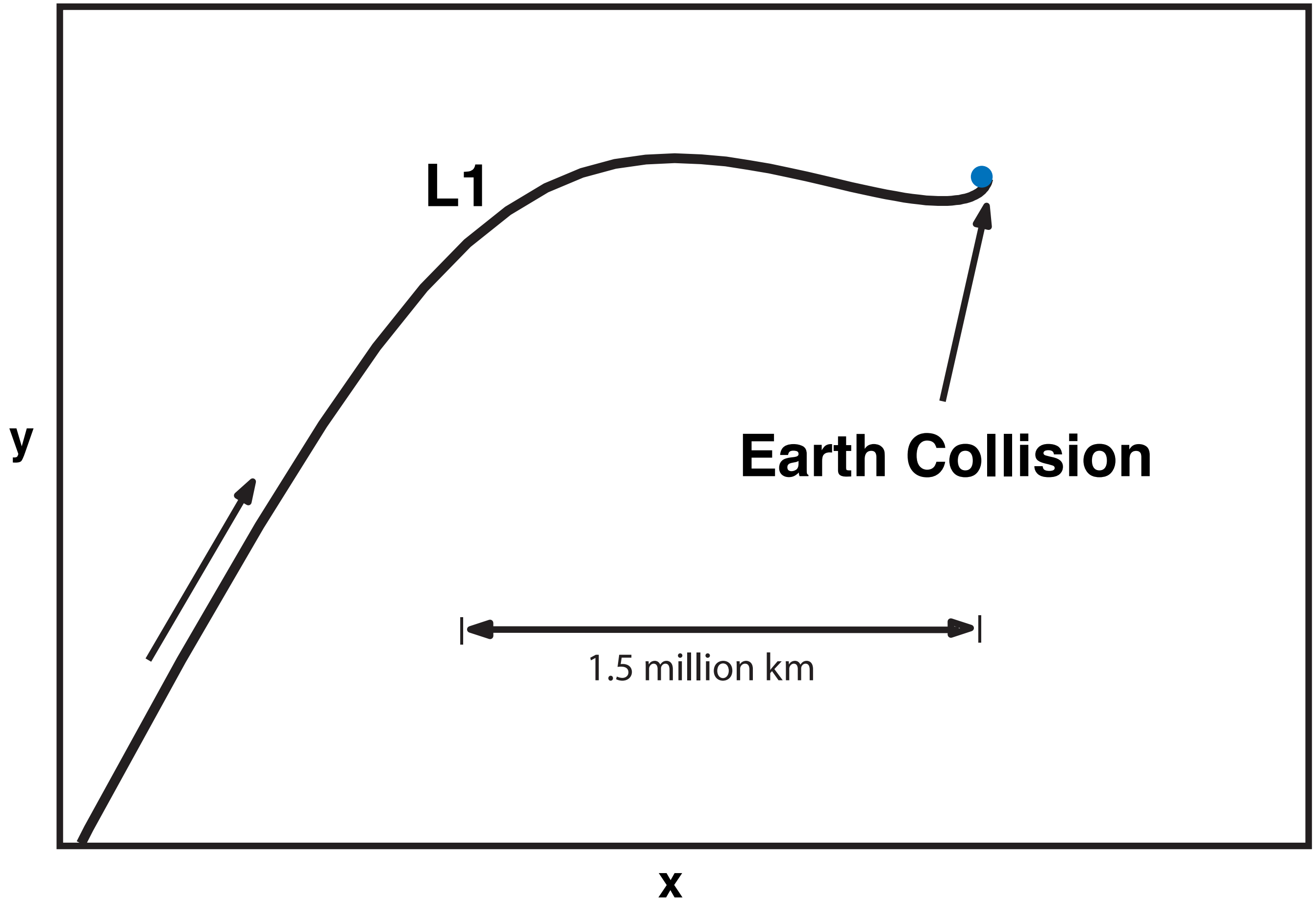
Probability for comet collision with Jupiter



Probability for NEA collision with Earth



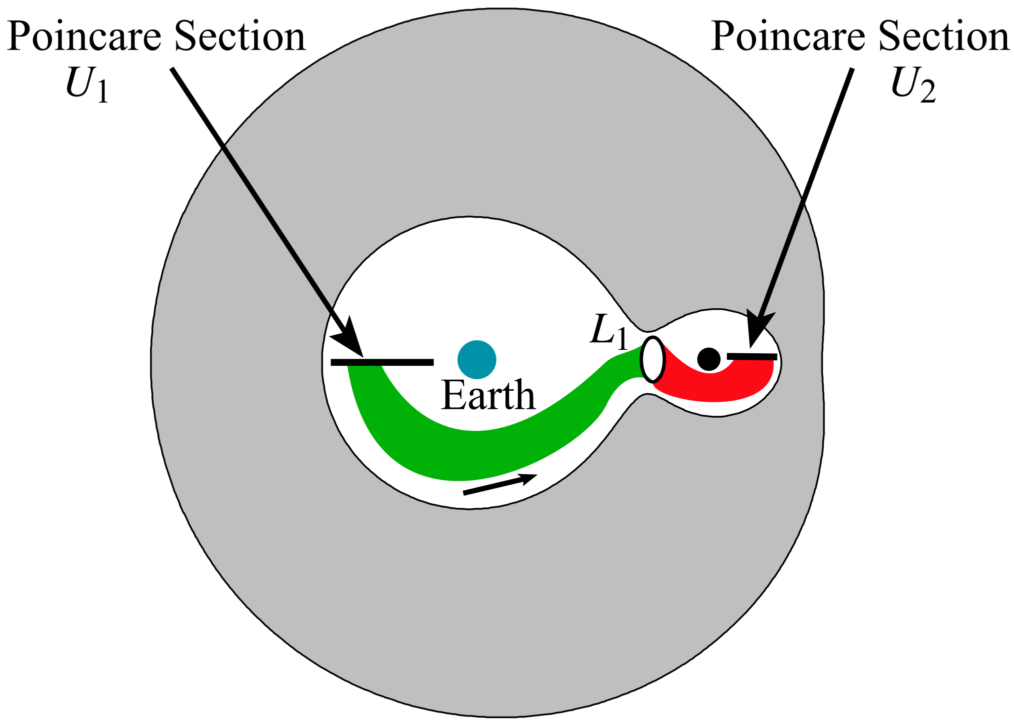
Typical collision orbit

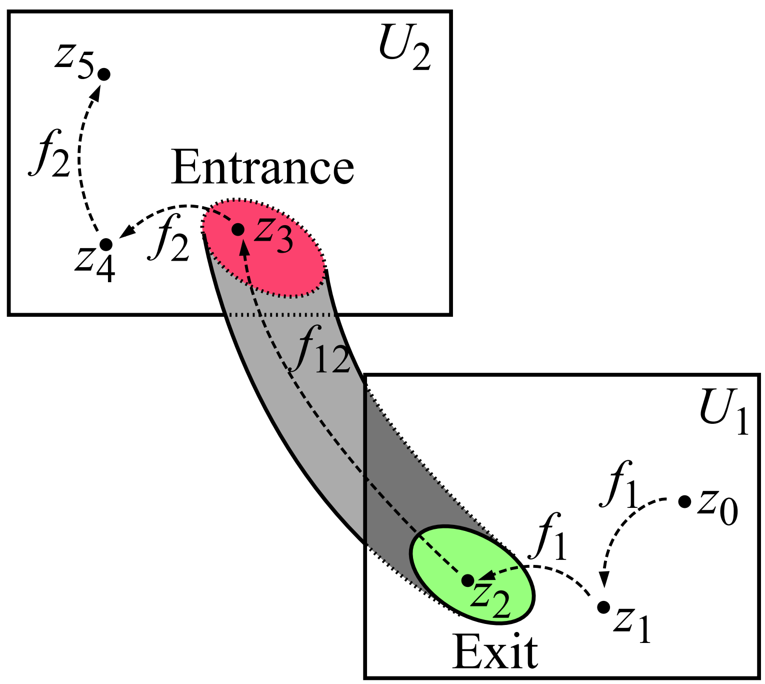


- Coming from direction of sun; harder to detect; **surprise!**

Multi-scale dynamics: Part 2

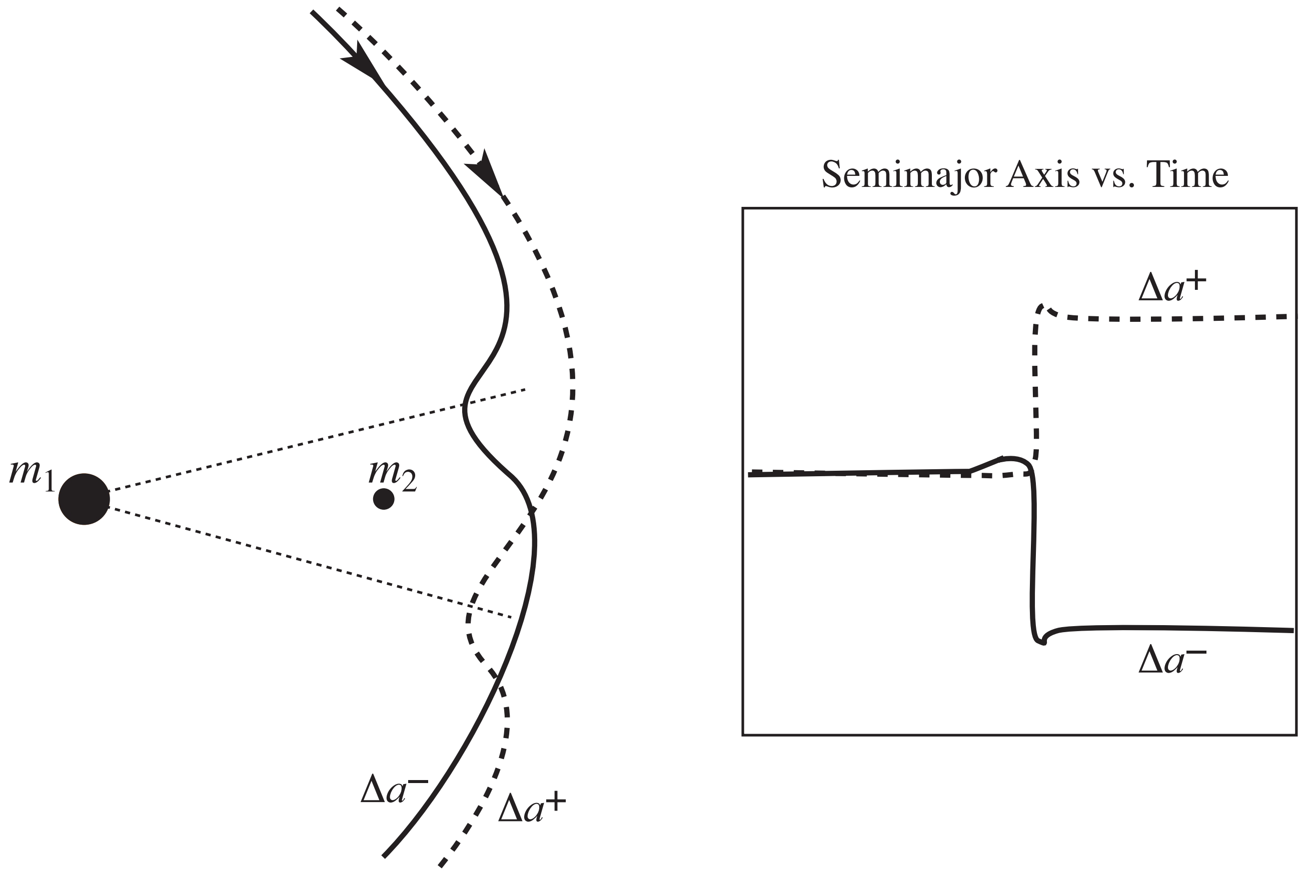
- Slices of energy surface: Poincaré sections U_i
- Tube dynamics: evolution **between** U_i
- What about evolution **on** U_i? $\longleftarrow$





Kicks at periapsis

- Key idea: model particle motion as "**kicks**" at periapsis

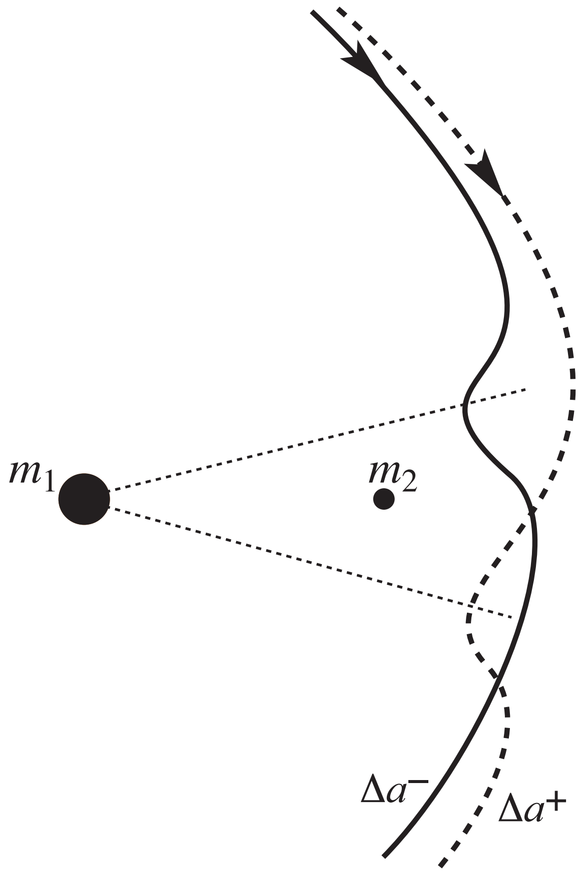


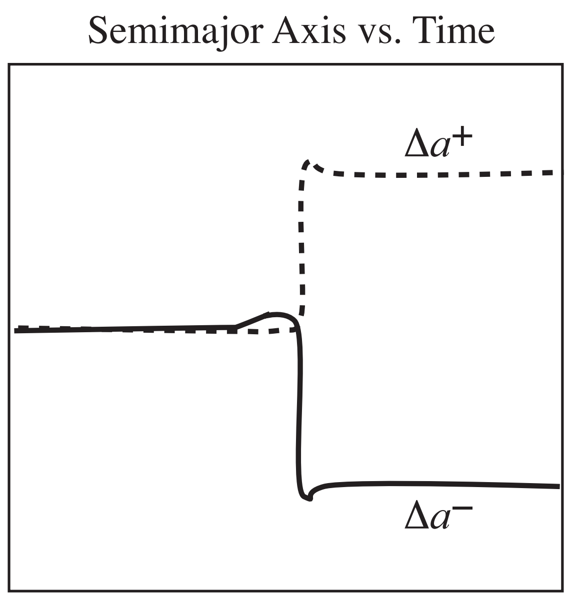
In rotating frame where m_1, m_2 are fixed

Ross & Scheeres, SIAM J. Appl. Dyn. Sys. [2007]

Kicks at periapsis

- Sensitive dependence on **argument of periapse** ω



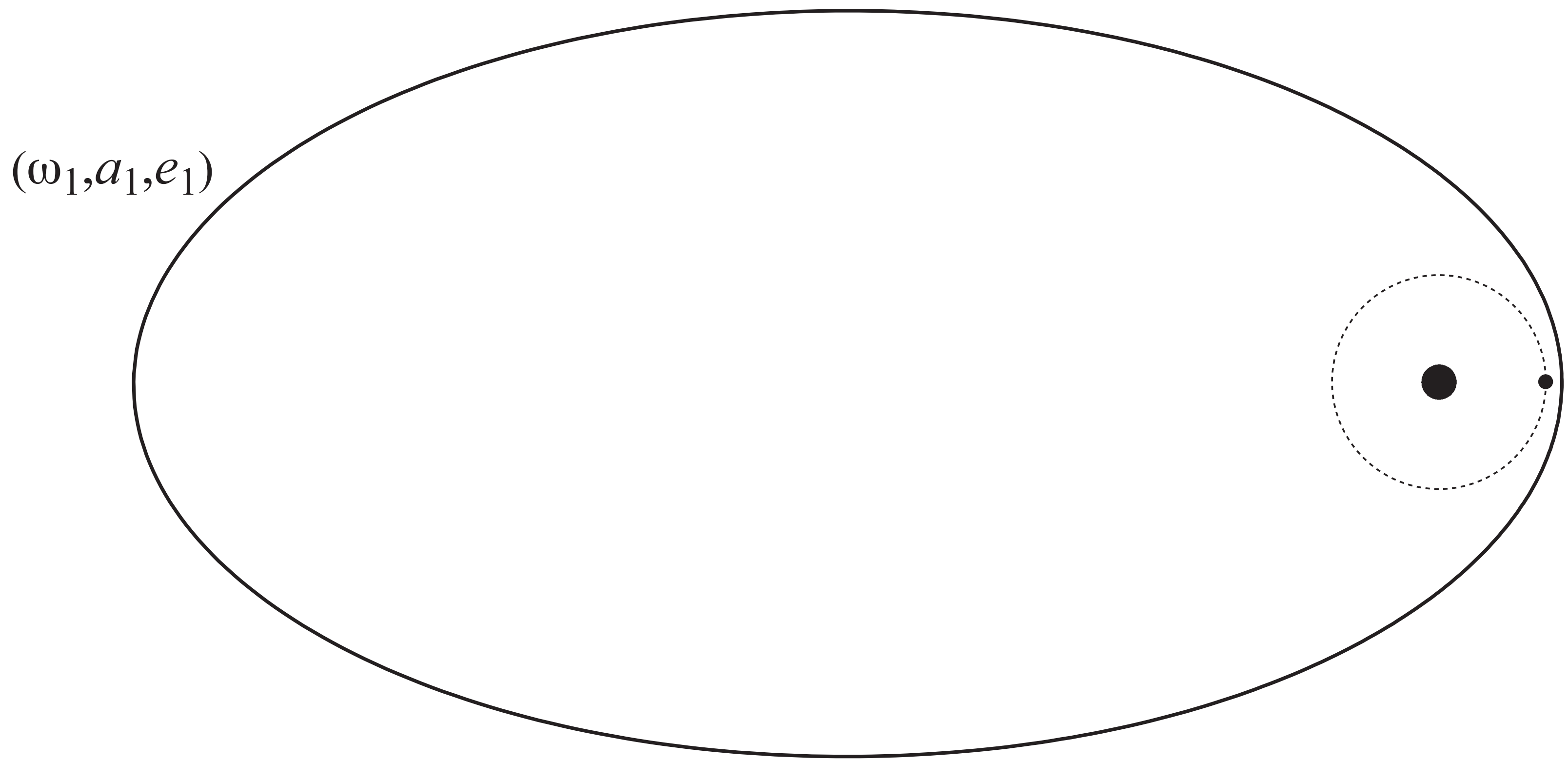


In rotating frame where m_1, m_2 are fixed

Ross & Scheeres, SIAM J. Appl. Dyn. Sys. [2007]

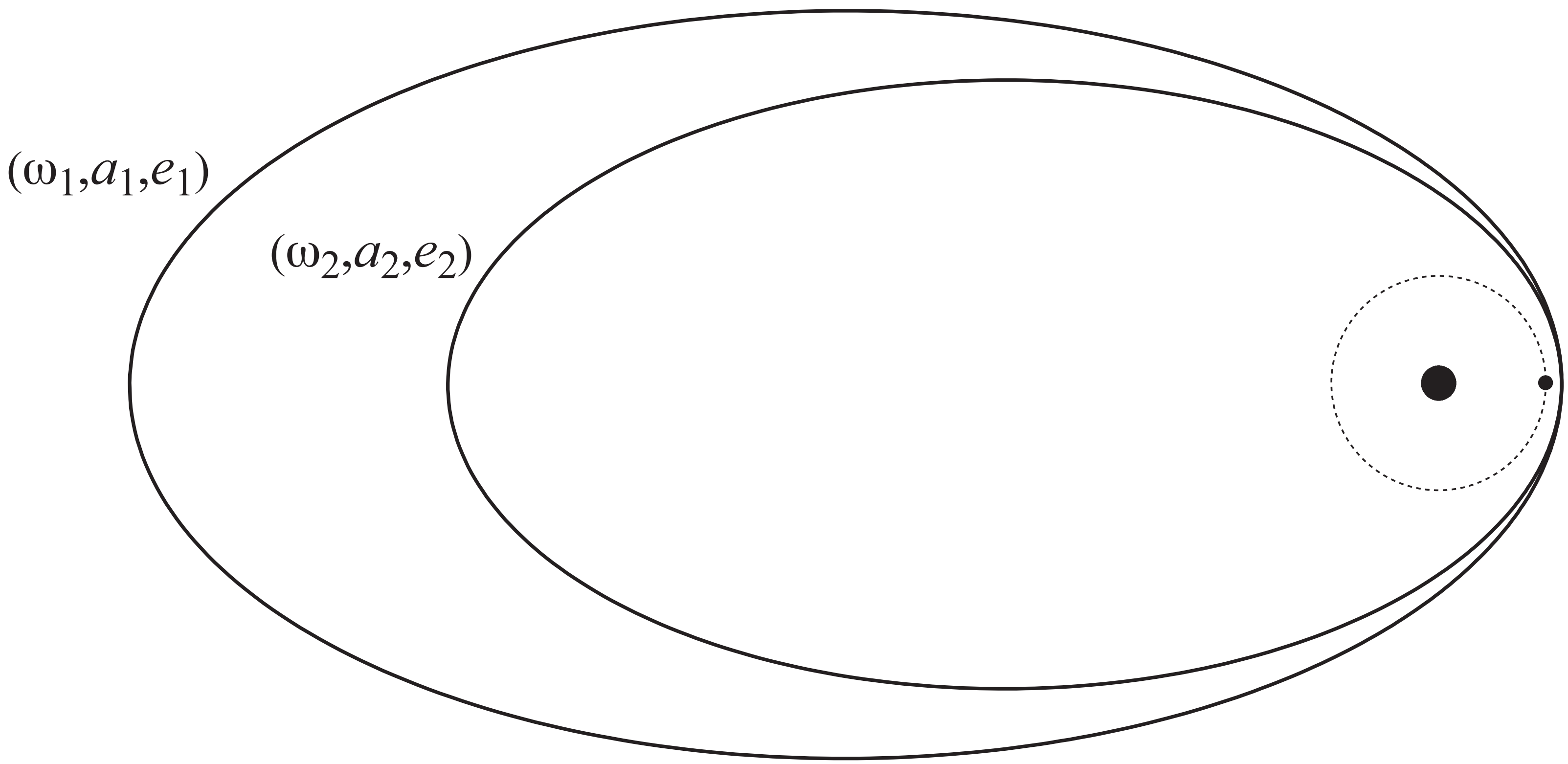
Kicks at periapsis

- Construct **update map** $(\omega_1, a_1, e_1) \mapsto (\omega_2, a_2, e_2)$ using average perturbation per orbit by smaller mass



Kicks at periapsis

- Construct **update map** $(\omega_1, a_1, e_1) \mapsto (\omega_2, a_2, e_2)$ using average perturbation per orbit by smaller mass



Nearly integrable Hamiltonian

- Particle assumed on **near-Keplerian orbit** around m_1
- Hamiltonian in nearly integrable action-angle form

$$H(I,\theta) = H_0(I) + \mu H_1(I,\theta), \quad \mu \ll 1,$$

i.e.,

$$H(L,G,l,\omega) = K(L) - G + \mu R(L,G,l,\omega)$$

in Delaunay (action-angle) variables

Change in orbital elements over one particle orbit

- Evolution of G (angular momentum)

$$\frac{dG}{dt} = -\mu \frac{\partial R}{\partial \omega},$$

- Approximate change in G over an orbit

$$\Delta G = -\mu \int_{\text{one orbit}} \frac{\partial R}{\partial \omega}\, dt$$

- $\Delta K =$ **Keplerian energy change** over an orbit

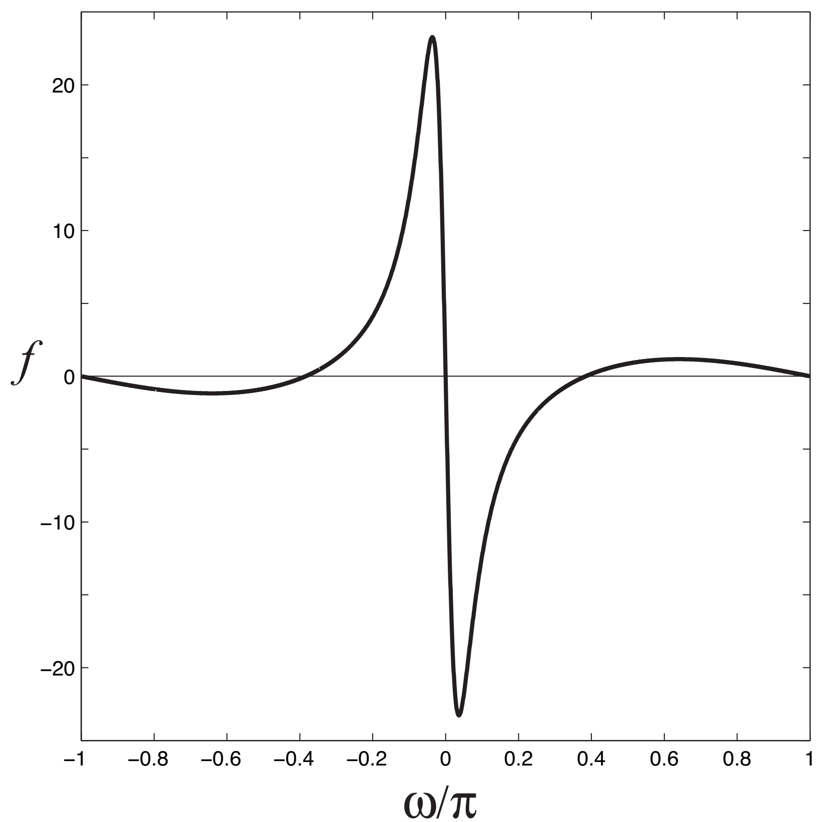
$$\Delta K = \Delta G - \mu \Delta R$$

Energy kick function

- Changes have form

$$\Delta K = \mu f(\omega),$$

f is the **energy kick function** with parameters K, E



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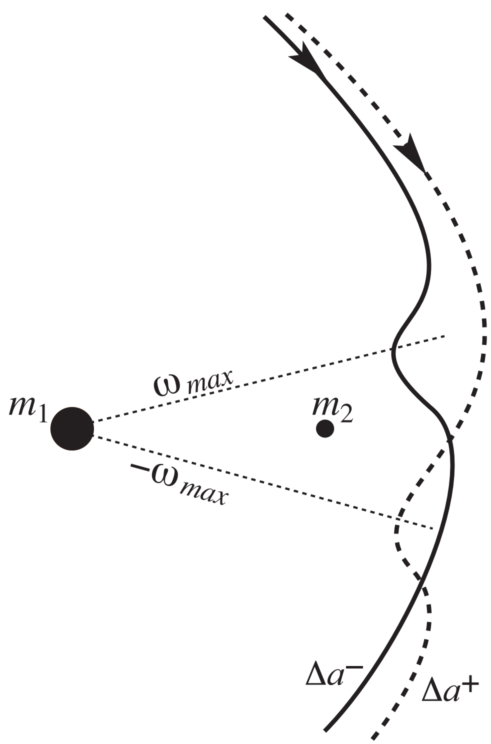
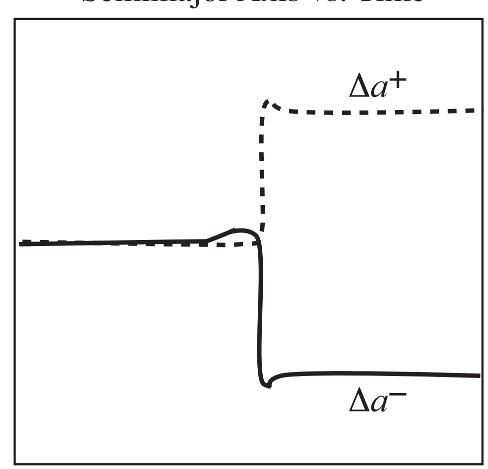




The periapsis kick map (Keplerian Map)

- Cumulative effect of **consecutive passes** by perturber

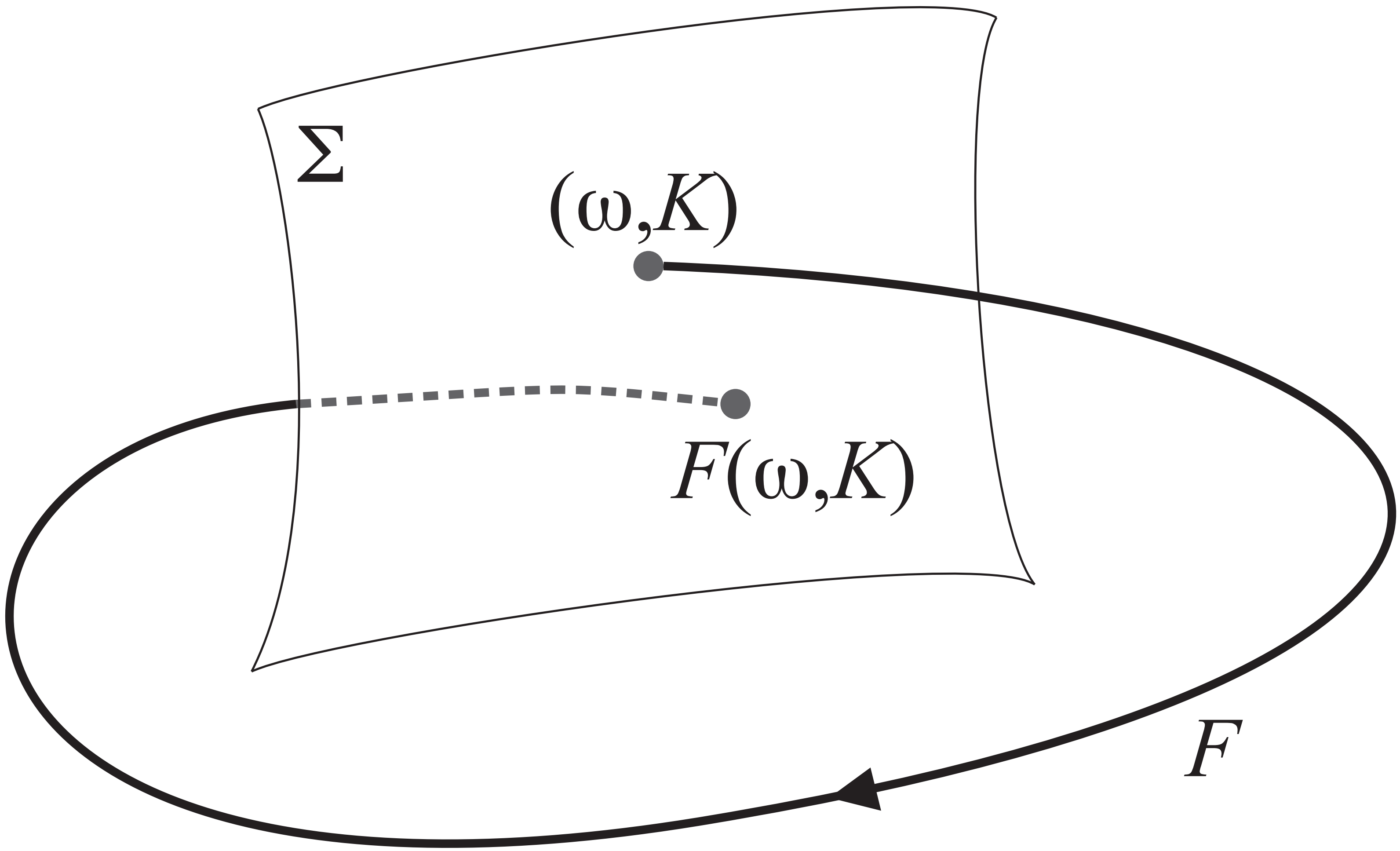
- Can construct an **update map** $(\omega_{n+1}, K_{n+1}) = F(\omega_n, K_n)$ on the cylinder $\Sigma = S^1 \times \mathbb{R}$, i.e., $F : \Sigma \to \Sigma$ where

$$\begin{pmatrix} \omega_{n+1} \\ K_{n+1} \end{pmatrix} = \begin{pmatrix} \omega_n - 2\pi(-2(K_n + \mu f(\omega_n)))^{-3/2} \\ K_n + \mu f(\omega_n) \end{pmatrix}$$

- **Area-preserving (symplectic twist) map**

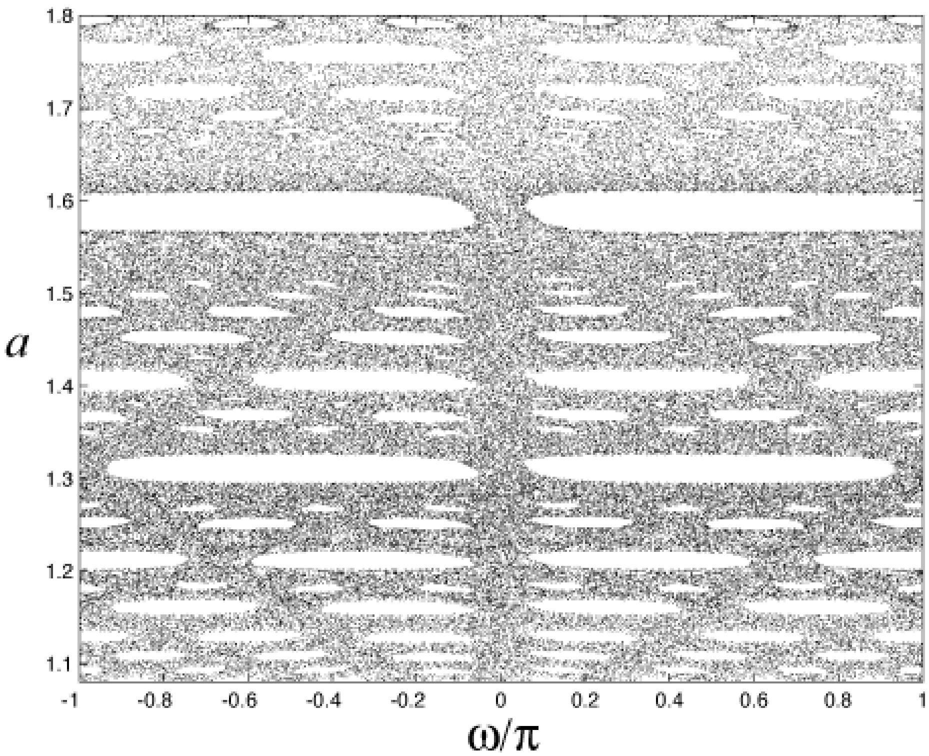
- Ex.: particle in Jupiter-Callisto system, $\mu = 5 \times 10^{-5}$

Identify Keplerian map as Poincaré return map



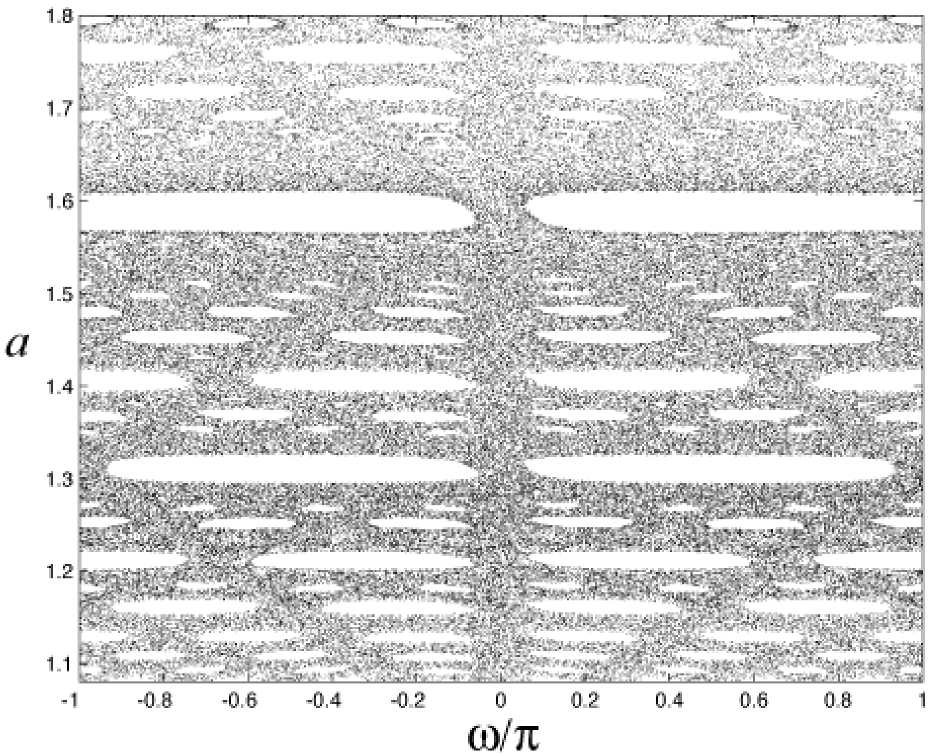
- **Poincaré map at periapsis** in orbital element space
- $F : \Sigma \to \Sigma$ where $\Sigma = \{l = 0 \mid H = E\}$

Verification of Keplerian map: phase portrait

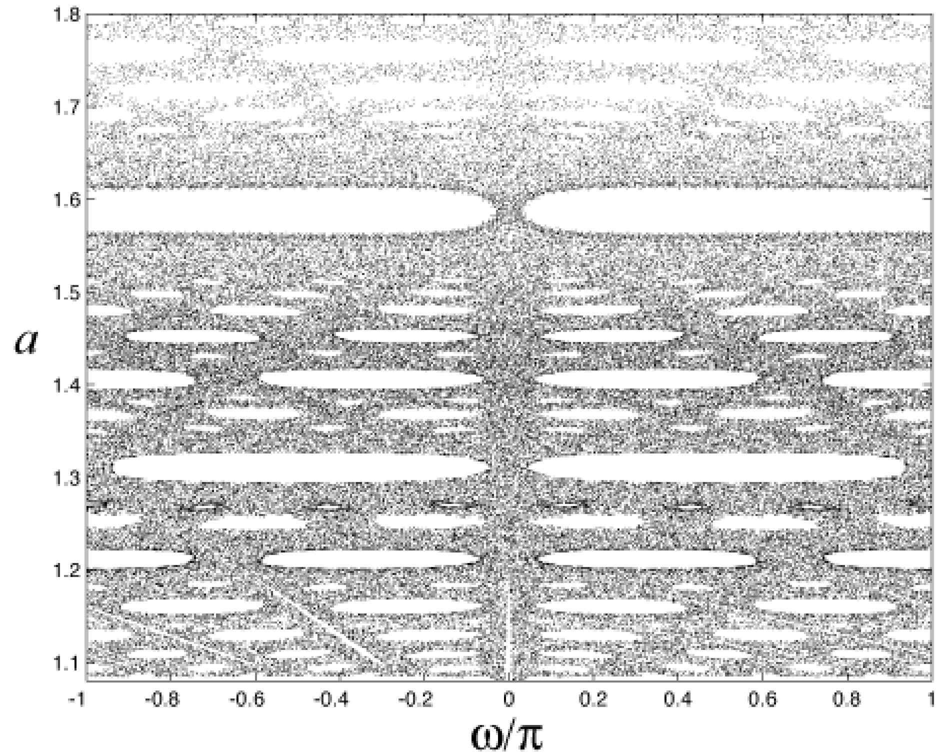


Keplerian map

Verification of Keplerian map: phase portrait



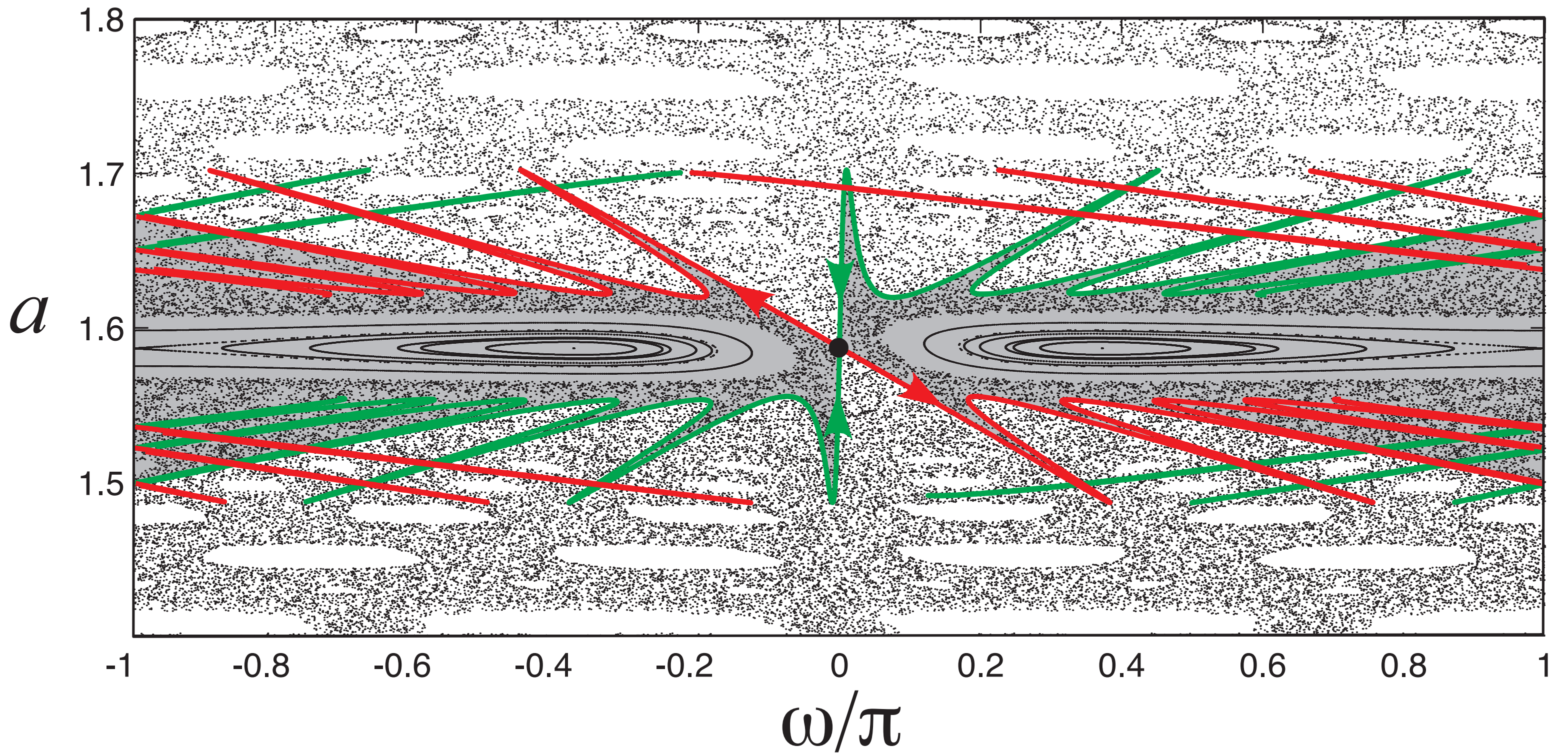
Keplerian map



numerical integration of full system

- Keplerian map = **fast orbit propagator**
- **preserves phase space features**
  — but breaks left-right symmetry present in original system
  — can be removed using another method (Hamilton-Jacobi)

Dynamics of Keplerian map

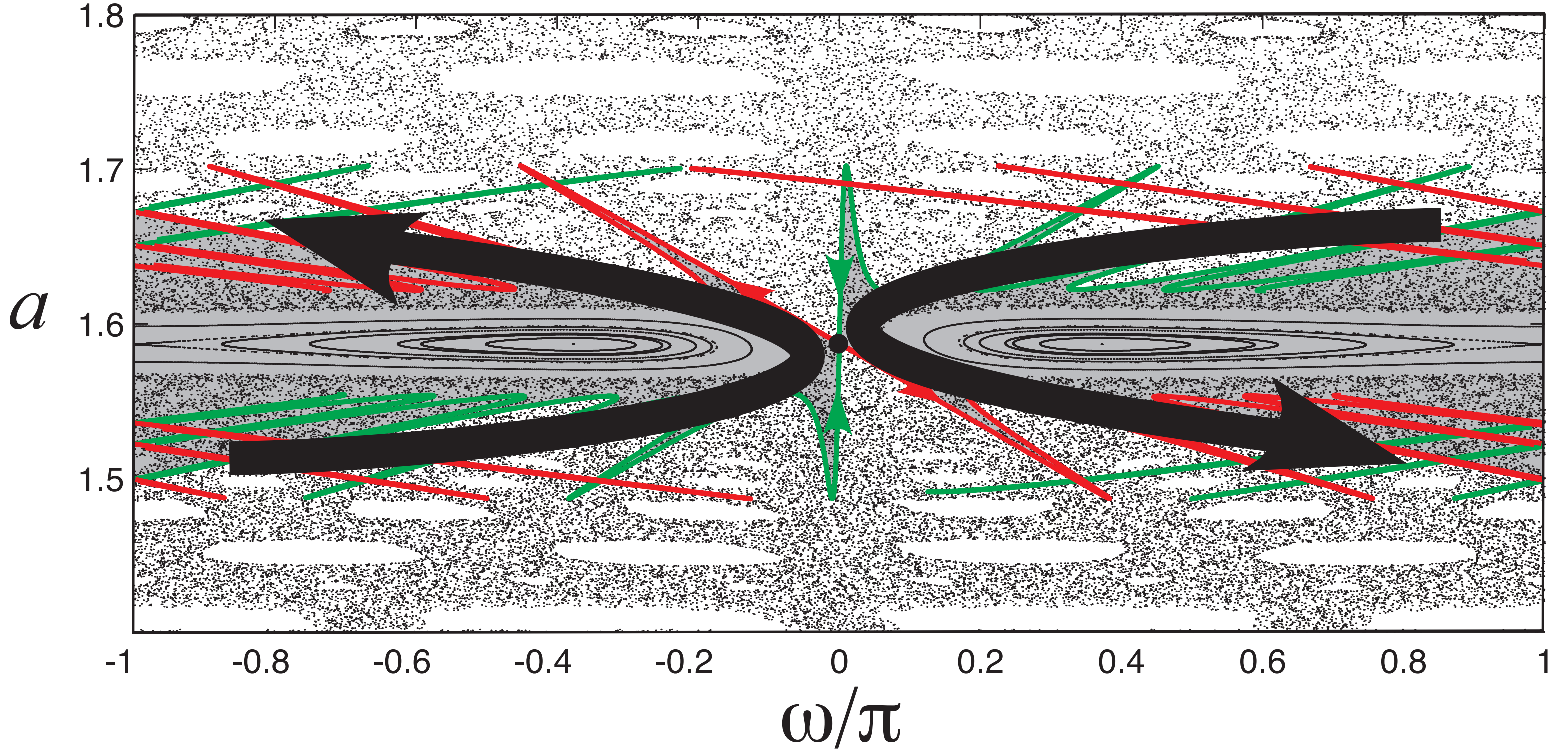


Resonance zone[3]

- ☐ Structured motion around resonance zones

[3] in the terminology of MacKay, Meiss, and Percival [1987]

Dynamics of Keplerian map



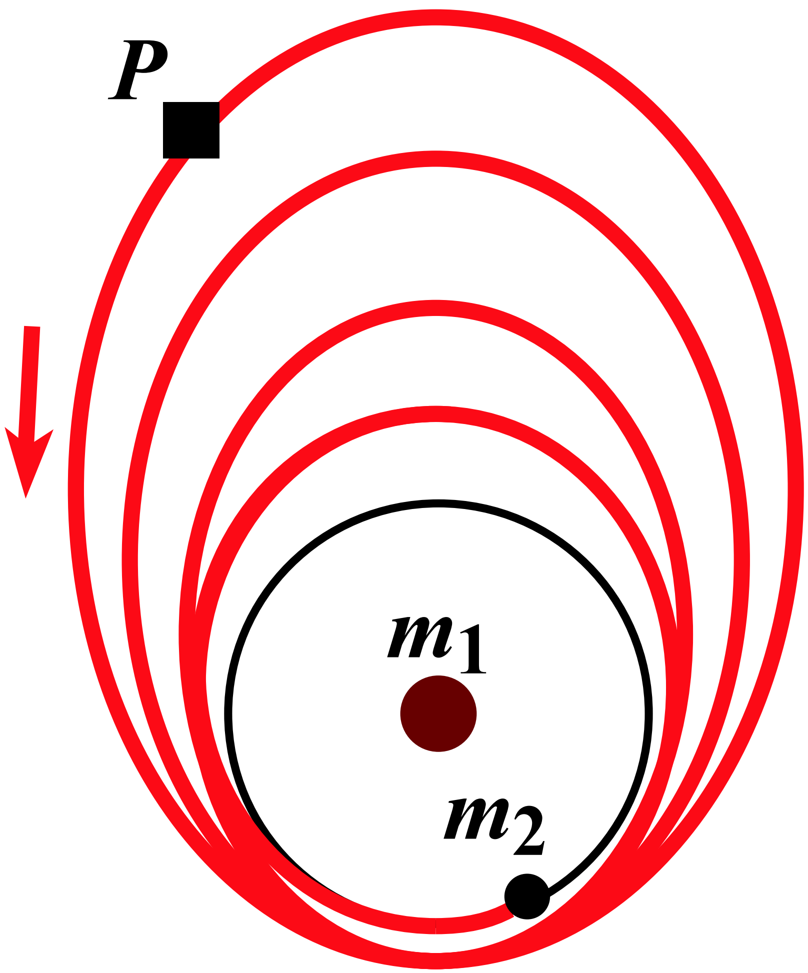
Resonance zone[4]

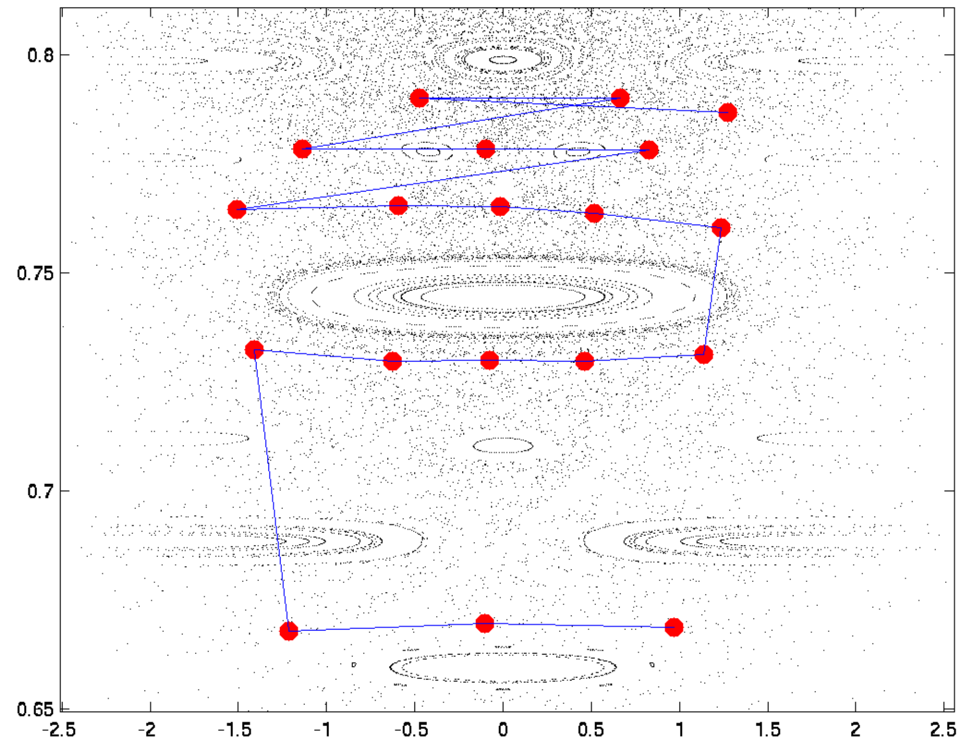
- ☐ Structured motion around resonance zones

[4] in the terminology of MacKay, Meiss, and Percival [1987]

Large orbit changes via multiple resonance zones

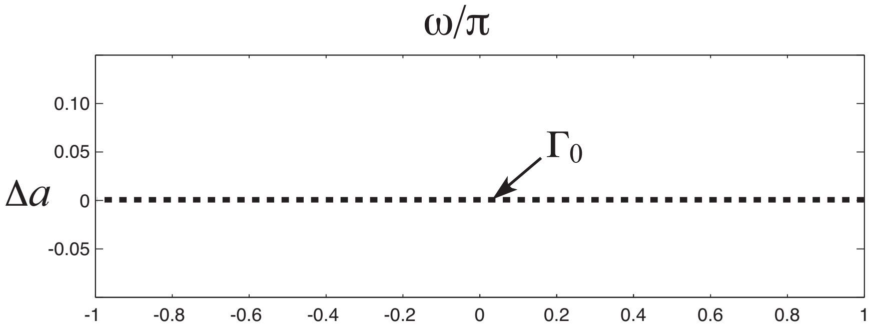
- **multiple flybys** for **orbit reduction or expansion**[5]



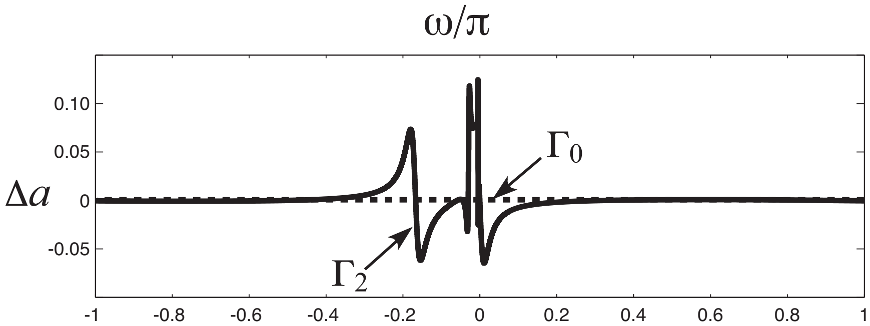


[5] Grover & Ross, J. Guid. Cont. Dyn. [2009]

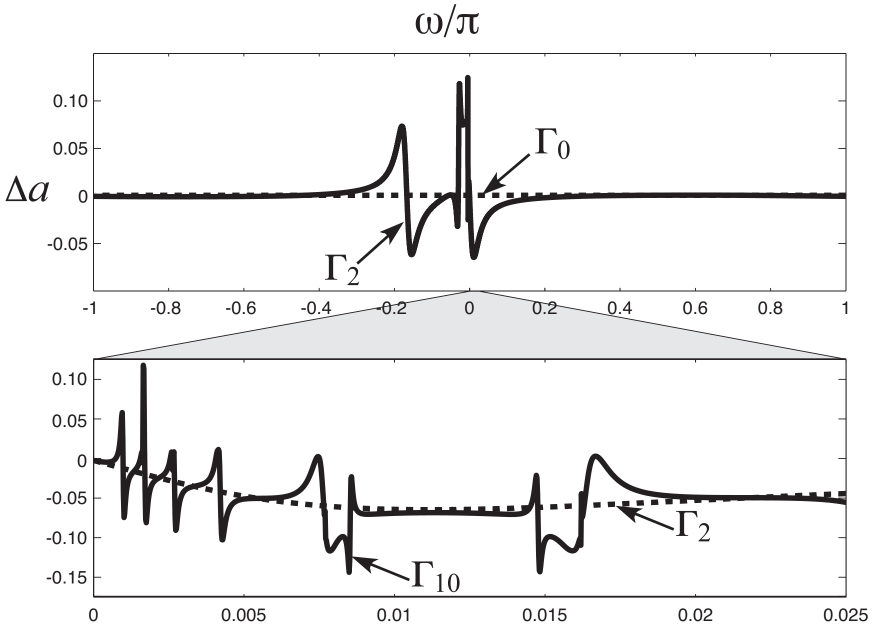
Large orbit changes, $\Gamma_n = F^n(\Gamma_0)$



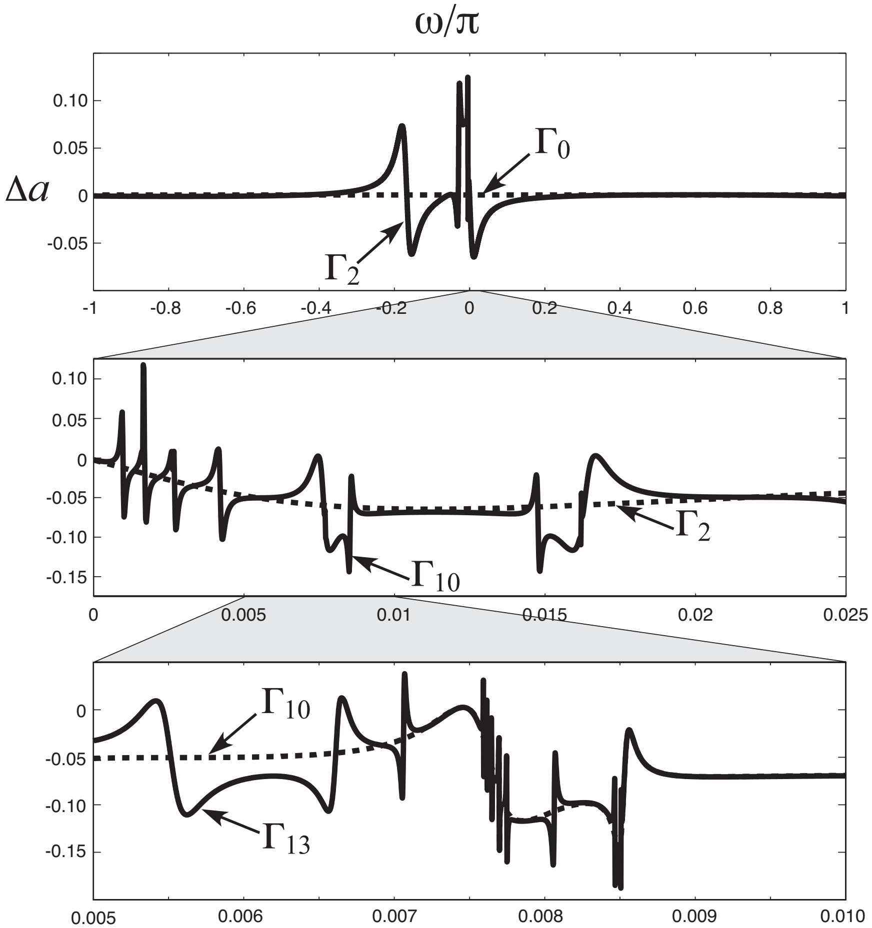
Large orbit changes, $\Gamma_n = F^n(\Gamma_0)$



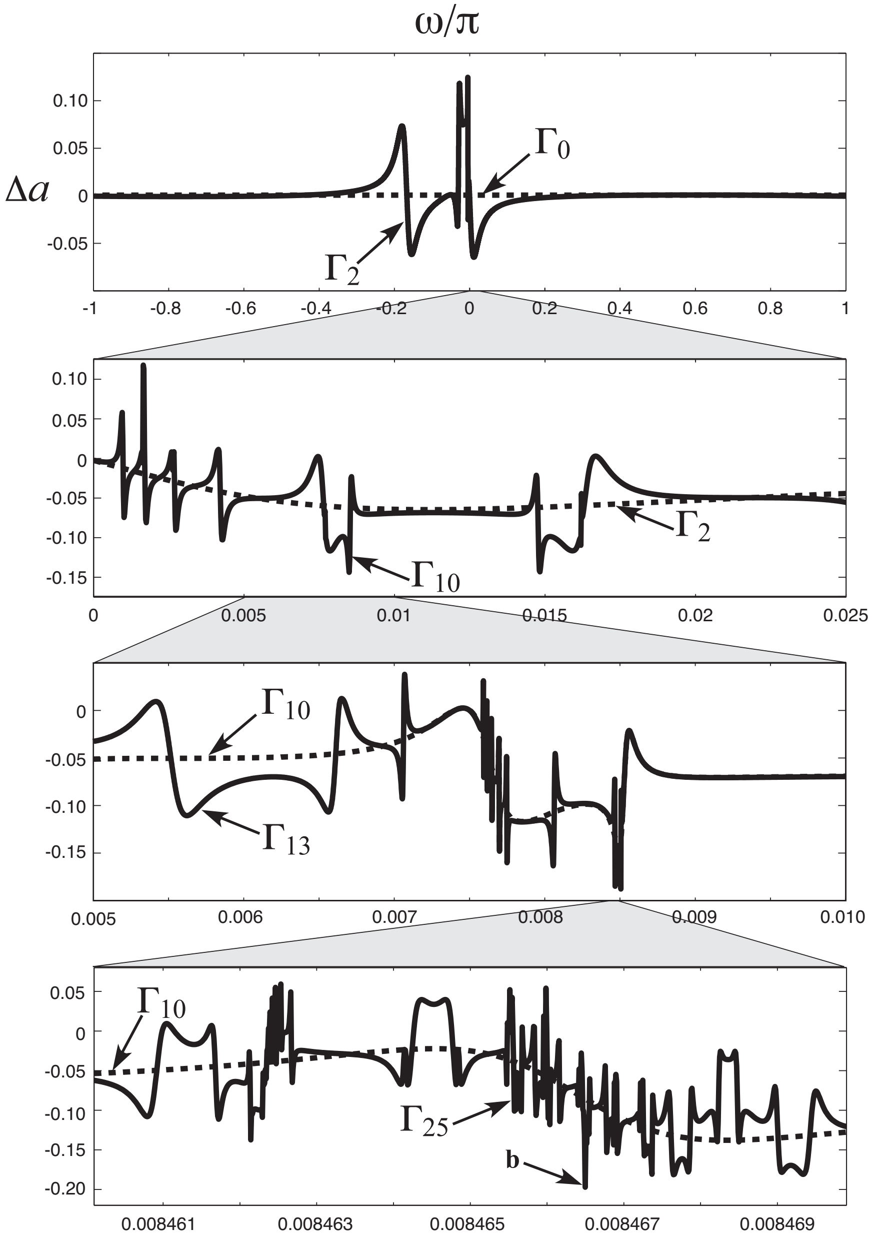
Large orbit changes, $\Gamma_n = F^n(\Gamma_0)$



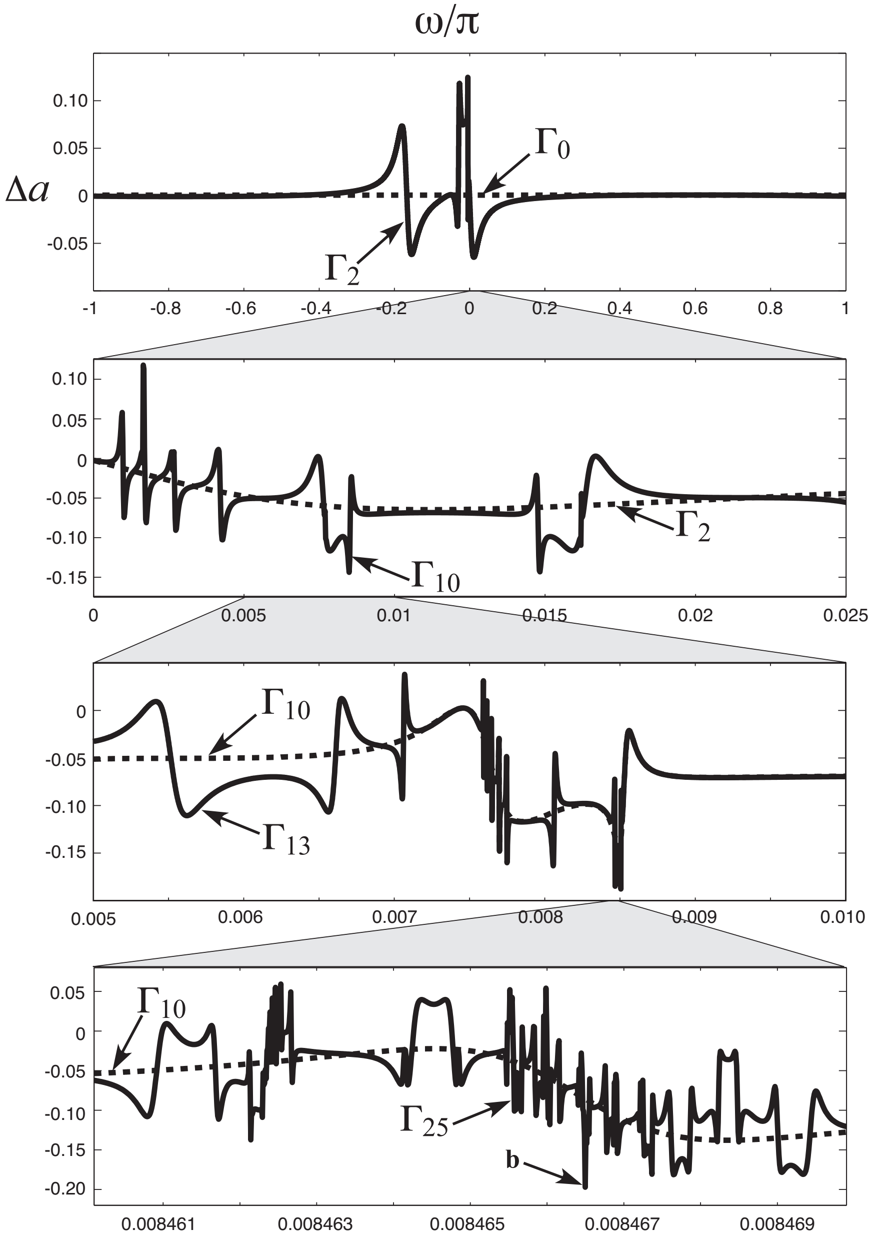
Large orbit changes, $\Gamma_n = F^n(\Gamma_0)$

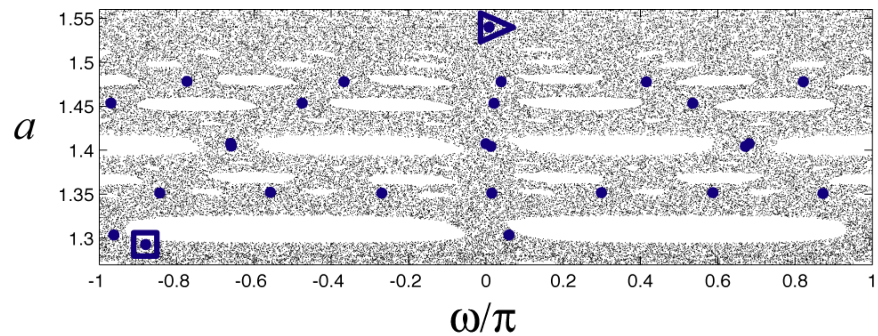


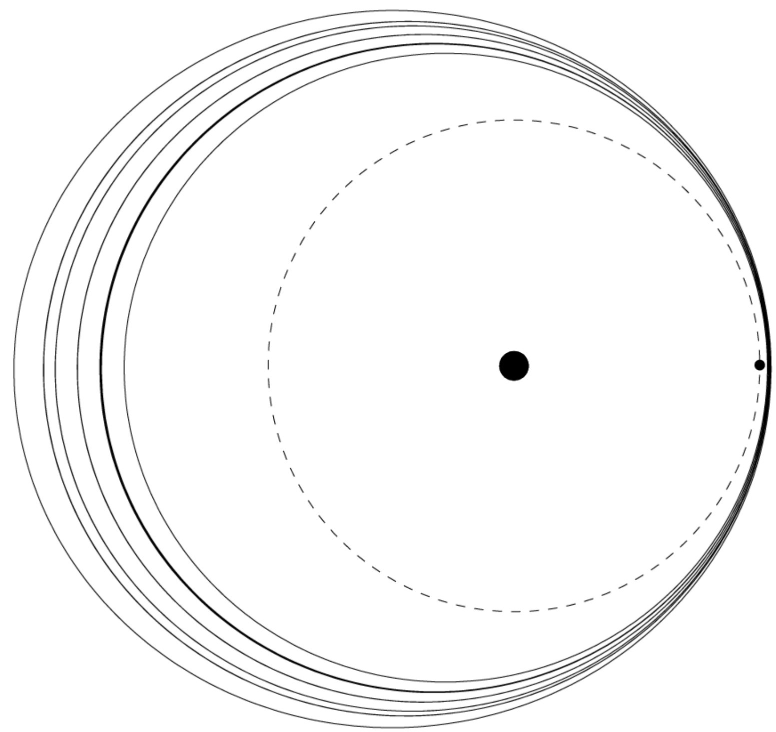
Large orbit changes, $\Gamma_n = F^n(\Gamma_0)$



Large orbit changes, $\Gamma_n = F^n(\Gamma_0)$

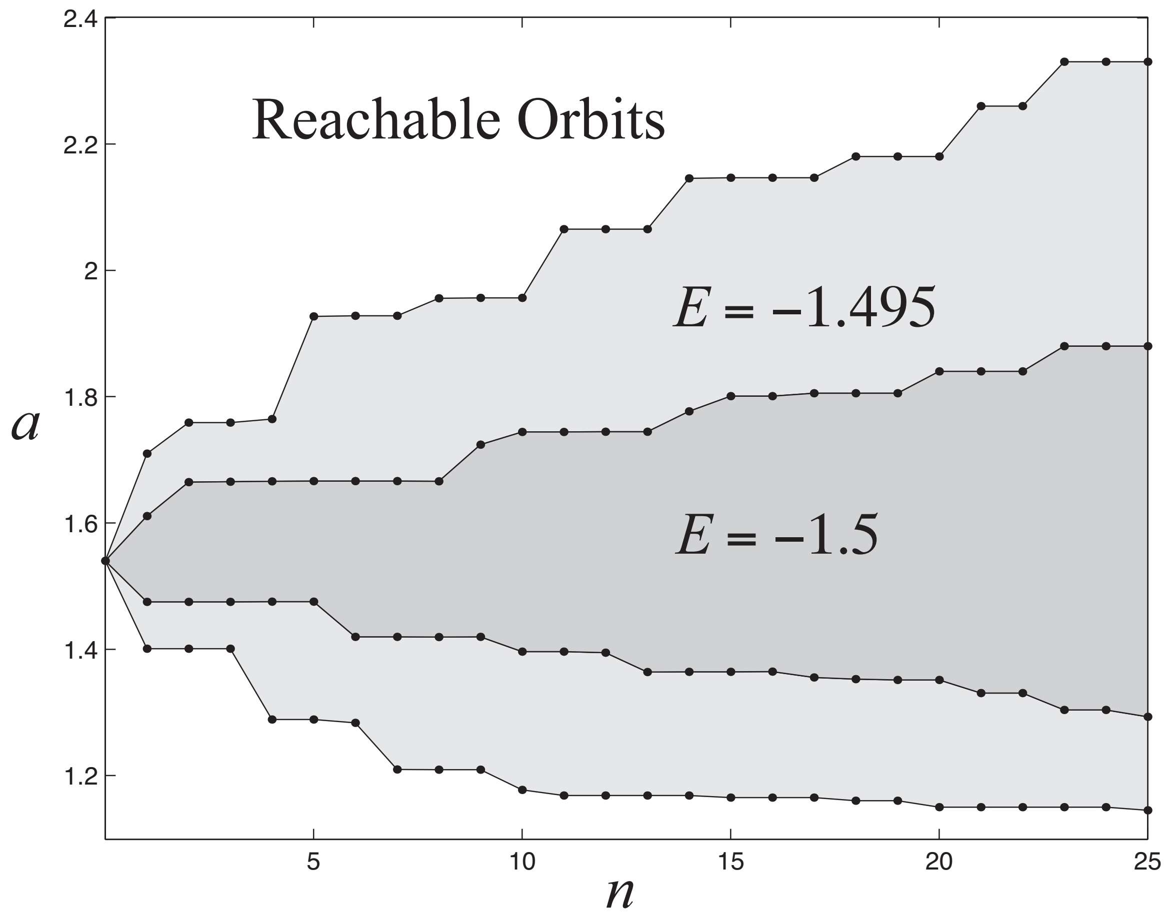






example trajectory

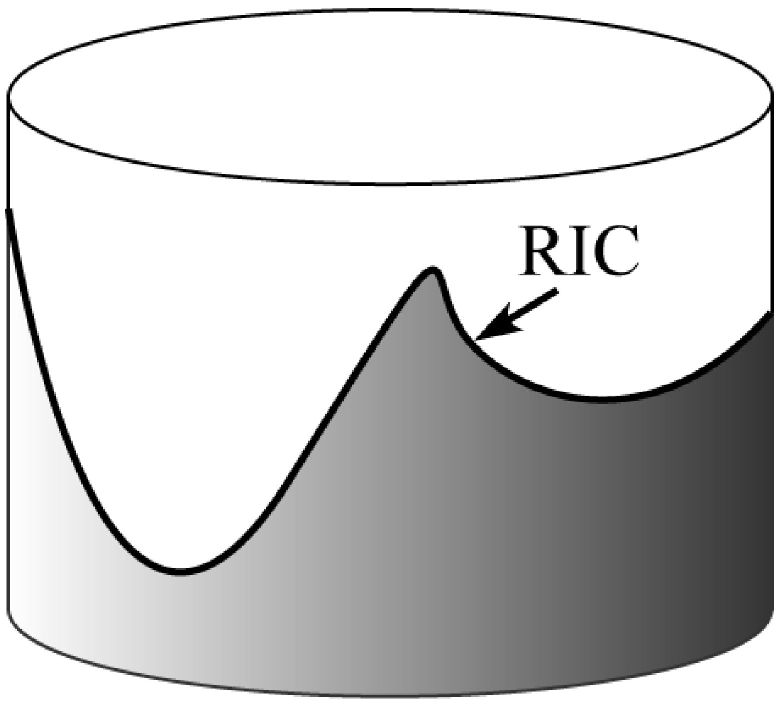
Reachable orbits and diffusion



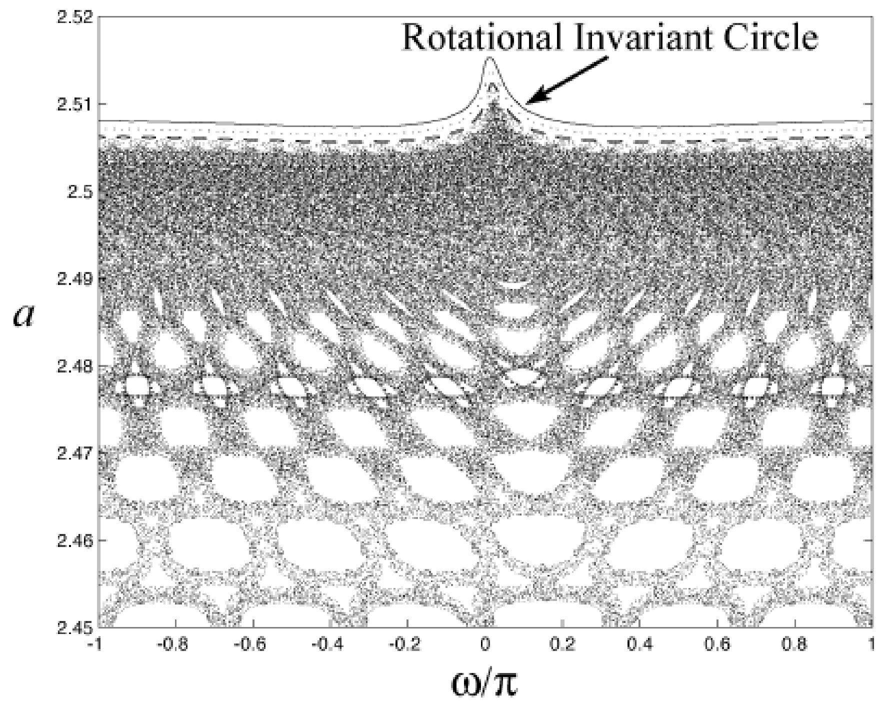
- Diffusion in semimajor axis

  … increases with E (larger kicks)

Reachable orbits: upper boundary for small μ

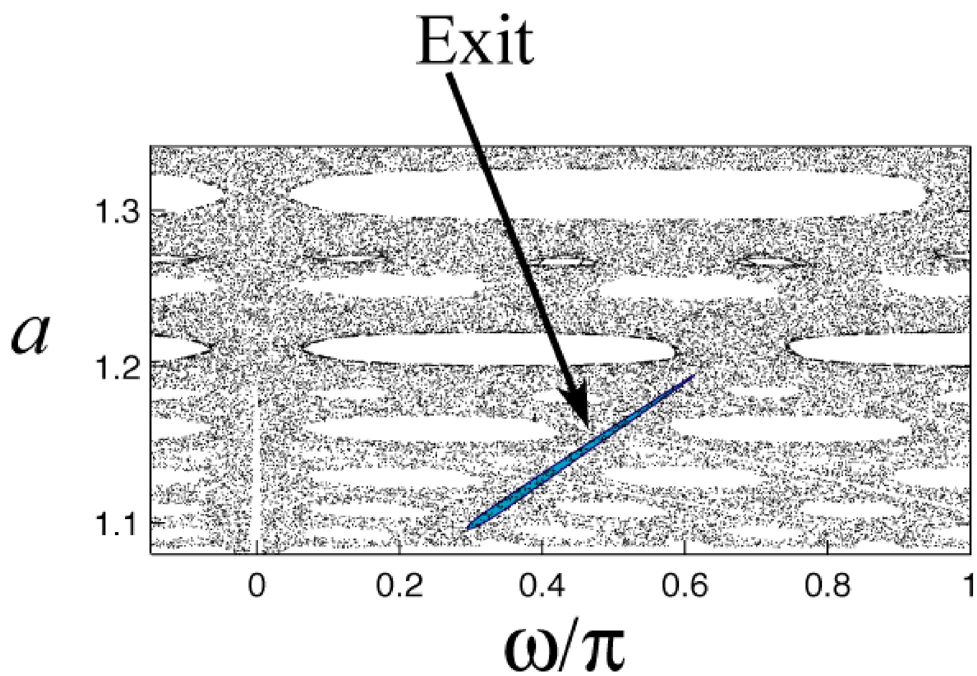


A rotational invariant circle (RIC)

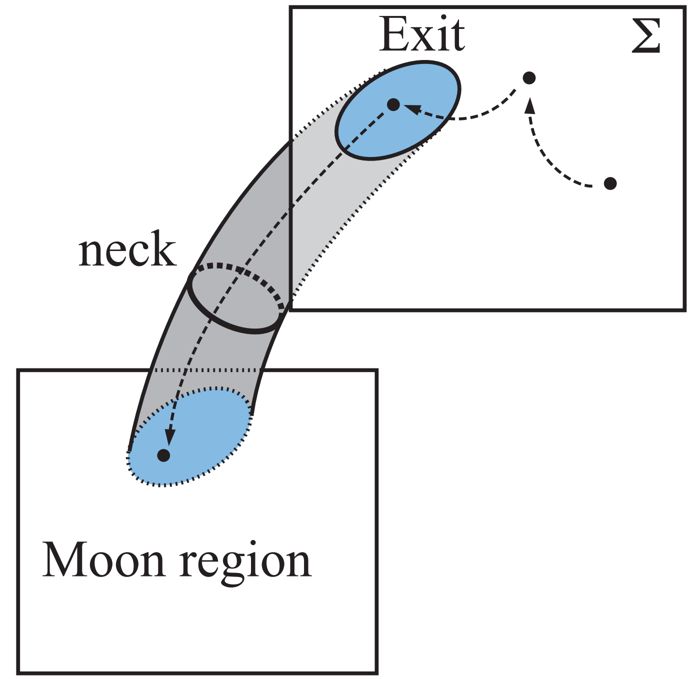


RIC found in Keplerian map for $\mu = 5 \times 10^{-6}$

Relationship to capture around perturber

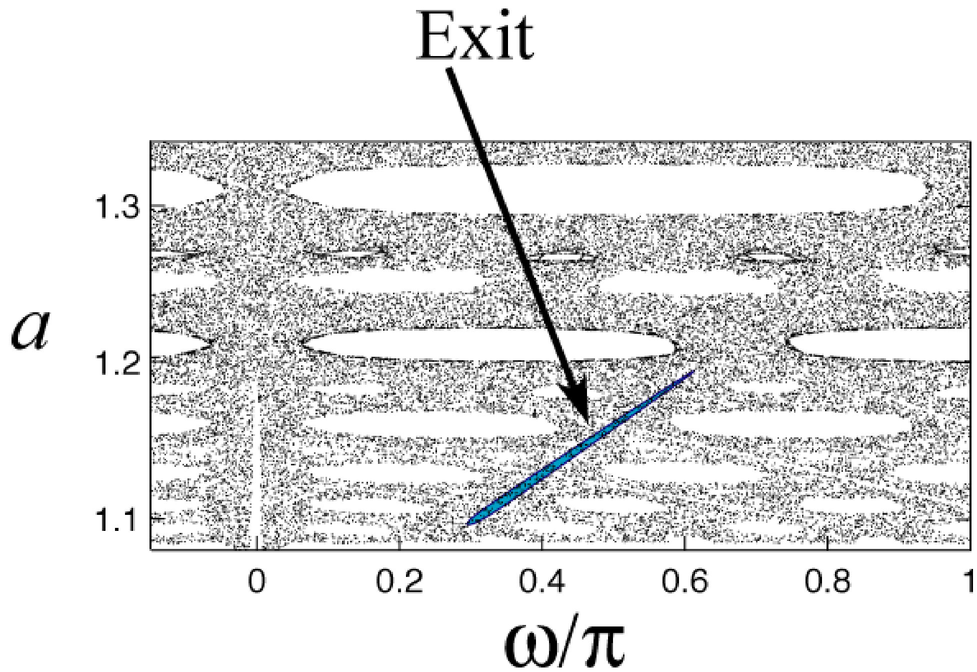


exit from jovicentric to moon region

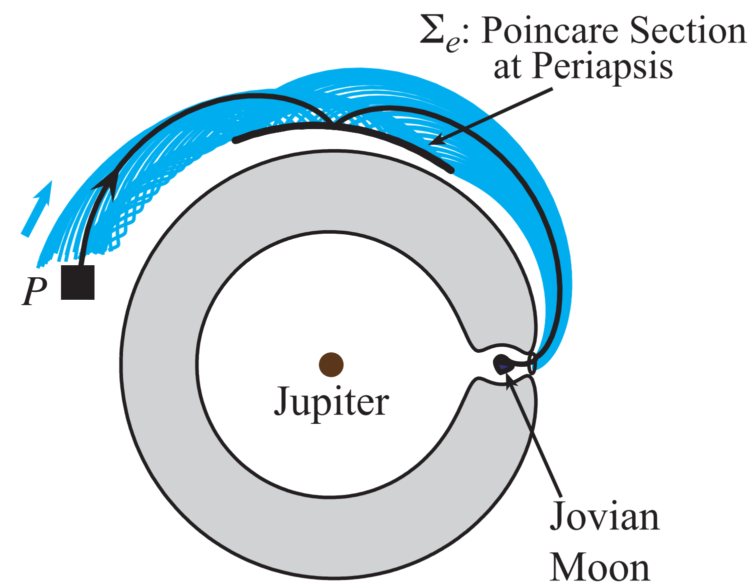


- **Exit**: where tube of capture orbits intersects Σ
- Orbits reaching exit are **ballistically captured**, passing by L_2

Relationship to capture around perturber



exit from jovicentric to moon region



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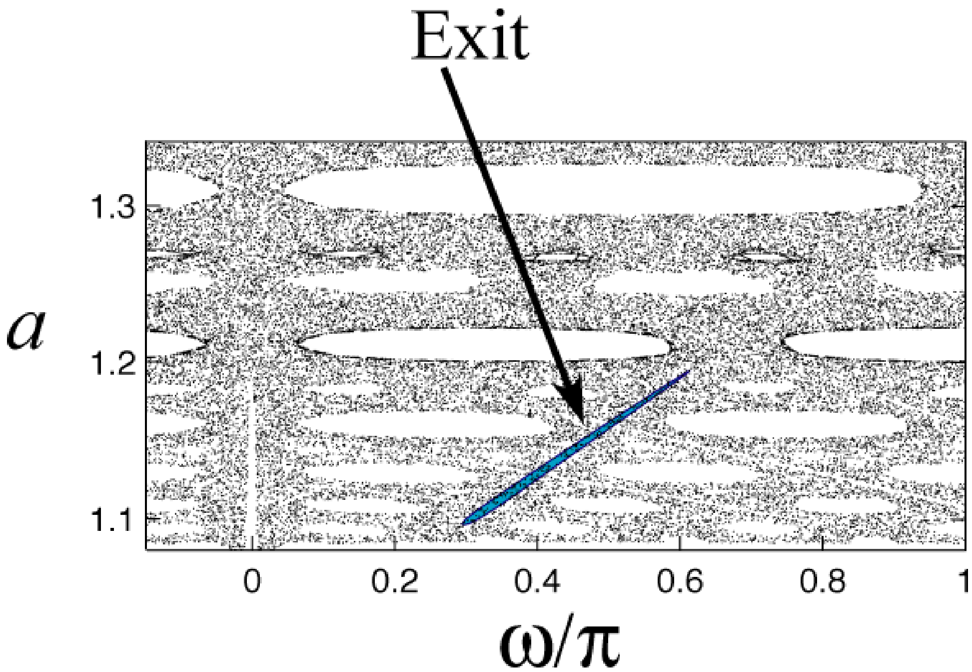
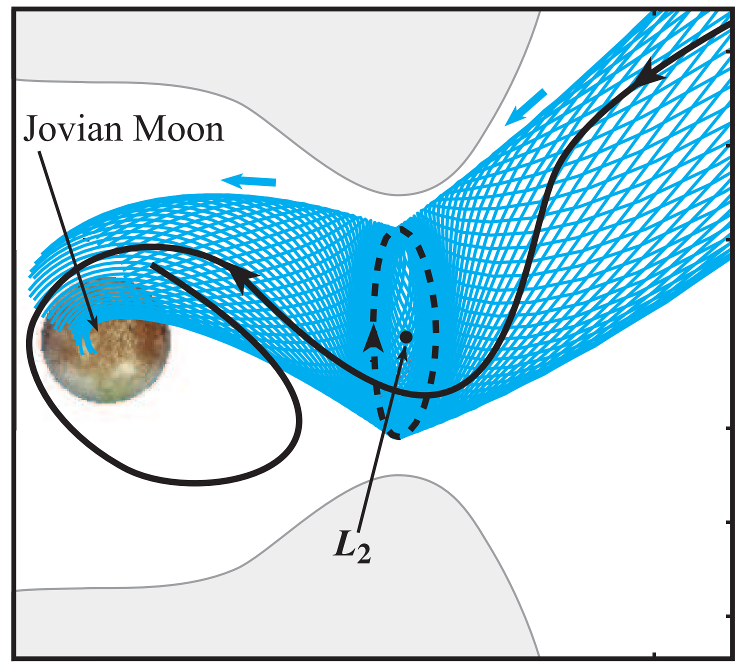
Relationship to capture around perturber

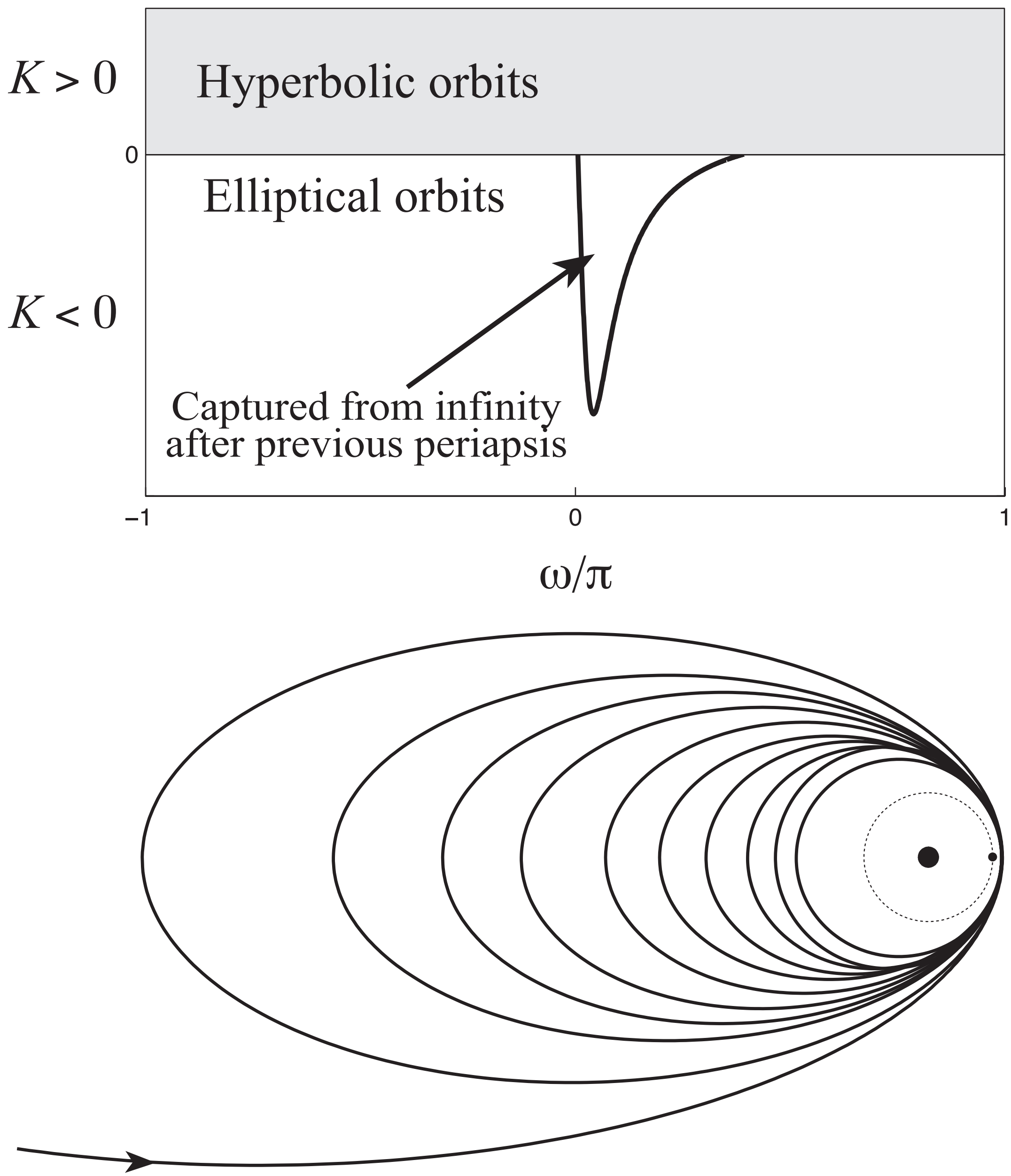




exit from jovicentric to moon region

- **Exit**: where tube of capture orbits intersects Σ
- Orbits reaching exit are **ballistically captured**, passing by L_2

Relationship to capture from infinity

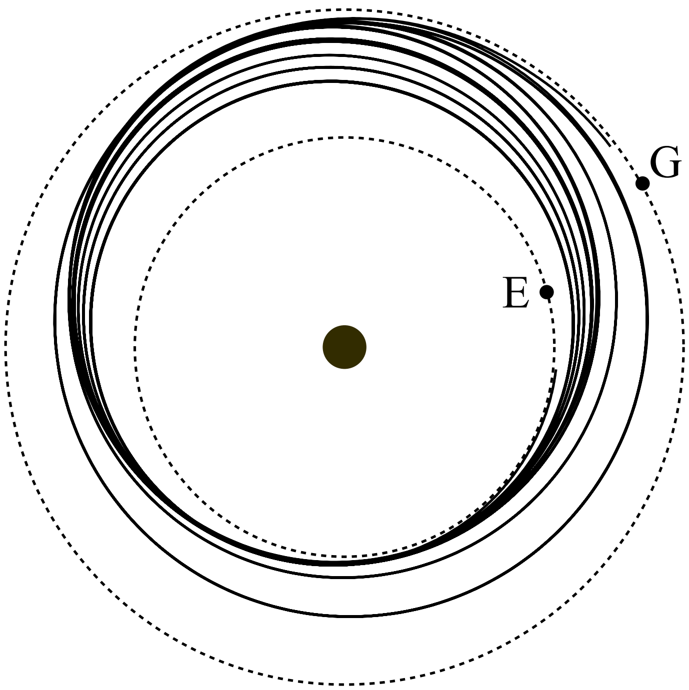


Jerg, Junge, Ross [2009]

Final word about Keplerian map

- Extensions:

  - out of plane motion (**4D map**)
  - control in the presence of uncertainty
  - eccentric orbits for the perturbers
  - multiple perturbers
    transfer from one body to another



- Consider other problems with spatially localized perturbations?
  – chemistry, vortex dynamics, ...

Conclusions

- **Invariant manifold tubes** are related to transport across rank 1 saddles (saddle $\times$ center $\times \cdots \times$ center)

- In the restricted 3-body problem:

- **Tube dynamics**: the interior of tube manifolds — related to capture, escape, transition, collision

- **Keplerian map** provides analytical expression approximating a Poincaré map

The End

Thank you!

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